MU120 Unit 15

The Open University

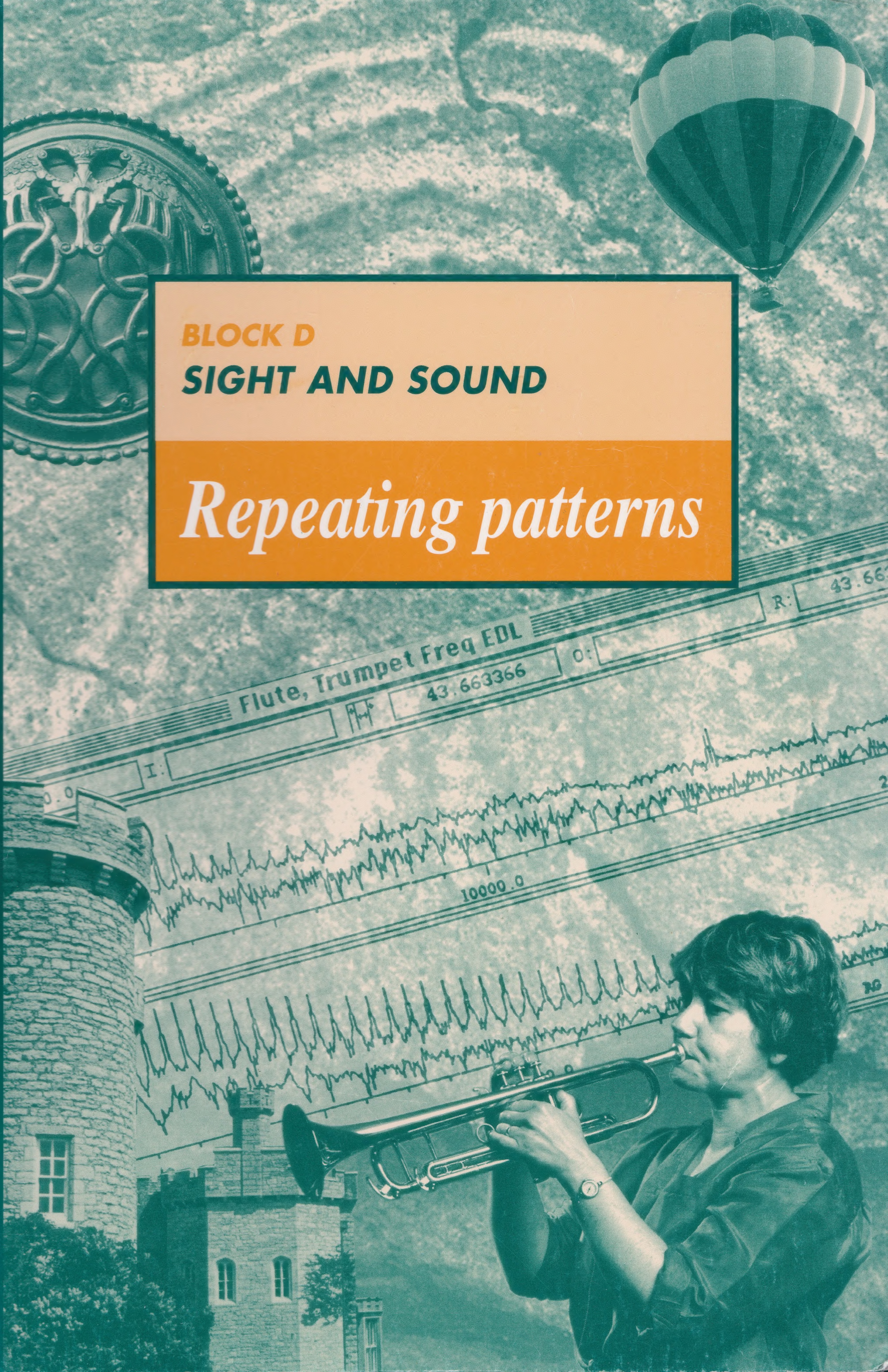
Mathematics and Computing
A first level multidisciplinary course

Open Mathematics

BLOCK D

SIGHT AND SOUND

Repeating patterns



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Prepared by the course team

MU120 course units were produced by the following team:

Mike Crampin (Author)
Margaret Crowe (Course Manager)
Fergus Daly (Academic Editor)
Chris Dillon (Author)
Judy Ekins (Chair and Author)
Barrie Galpin (Author)
Alan Graham (Author)
Linda Hodgkinson (Author)
Gillian Iossif (Author)
Eric Love (Author and Academic Editor)
David Pimm (Author and main Academic Editor)

Other contributions to the text were made by a number of Open University staff and students and others acting as consultants, developmental testers, critical readers and writers of draft material. The course team are extremely grateful for their time and effort.

The course units were put into production by the following:

Course Materials Production Unit (Faculty of Mathematics and Computing)

Martin Brazier (Graphic Designer)
Hannah Brunt (Graphic Designer)
Alison Cadle (TEXOpS Manager)
Jenny Chalmers (Publishing Editor)
Sue Dobson (Graphic Artist)
Roger Lowry (Publishing Editor)
Diane Mole (Graphic Designer)
Nazlin Vohra (Graphic Designer)

The Open University, Walton Hall, Milton Keynes, MK7 6AA.

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1.4

Contents

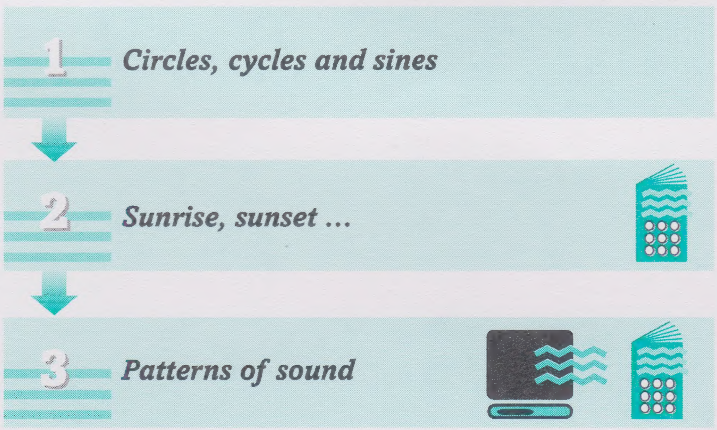
Study guide

This unit has three sections. It focuses on using sine curves to model periodic behaviour. The unit is intended to consolidate earlier work on the sine function in *Units 9*, *13* and *14*, and to provide more discussion of mathematical modelling.

Section 1 reviews basic mathematical ideas associated with sines and their relationship to motion in a circle. Section 2 looks at a modelling example in which a sine curve is fitted to a set of data.

Section 3 is based on the video band 'The sound of silence' followed by a shorter piece 'Subtractin sound'. The video lasts about 20 minutes. You should watch the video all the way through and work on the associated activity. Do not worry if you do not understand everything in the video at first. The ideas come up again in the *Calculator Book*. If you have time, watch the video again when you are tackling Activity 30. At the end of the section you will be asked to complete a Learning File and Handbook activity. Make sure that you add new mathematical ideas and formulas to your handbook as you work through the unit, and consolidate existing entries on trigonometric functions in your library of functions.

You should find that this unit consolidates ideas about modelling that you have met earlier in the course. Use this opportunity to make sure that your handbook is up to date, and that you add further notes on the mathematical modelling discussed in this unit.



Summary of sections and other course components needed for *Unit 15*

Introduction

> What the mathematician does is examine abstract 'patterns'—numerical patterns, patterns of shape, patterns of motion, patterns of behaviour, and so on. Those patterns can be either real or imagined, visual or mental, static or dynamic, qualitative or quantitative, purely utilitarian or of little more than recreational interest. They can arise from the world around us, from the depths of space and time, or from the inner workings of the human mind.
>
> (Keith Devlin (1994) *Mathematics: the science of patterns*, Scientific American Library, New York, p. 3)

Repetitive events lie at the very root of human existence. Electrical brain rhythms, heart beats, sleeping and waking patterns, and menstrual cycles are some of the regular biological activities that govern people's lives. Outside our bodies the relative periodic movements of the Earth, Sun and Moon determine the tides, the length of day and night, the monthly lunar cycle, the seasons and the length of a year.

Repeating patterns such as these form the backdrop against which people live their lives and, arguably, influence the way we think. From a mathematician's point of view, seeing the world in terms of patterns offers a way of making sense of complicated behaviour by thinking in terms of simpler mathematical 'building blocks' put together according to particular rules.

This unit focuses on the sine curve—a mathematical building block for a wide range of periodic behaviour. You have already met the sine curve in *Unit 9*, where it was used to model a steady musical tone from a tuning fork. You also saw in Chapter 9 of the *Calculator Book* how sine waves could be added together to produce other regular waveforms which were not themselves sine curves but which repeated periodically. In this unit, you are going to explore the mathematics of the sine curve further, and use the calculator to display the features of these special periodic shapes.

Recall that a sine curve is the graph of a periodic mathematical function, one that repeats its same basic shape over and over again.

The Learning File work focuses on the decisions you make as you work through the unit and what you base those decisions on.

1 Circles, cycles and sines

Aims The main aim of this section is to review some of the mathematical language used to talk about sine curves and periodic behaviour. ◇

You first met mathematical sine curves in *Unit 9* where they were used as models of pure musical notes. In *Unit 14*, you saw that there was a relationship between circles, sines and cosines. Later in this unit, you will see that sums of sine and cosine functions offer a way of describing complex periodic behaviour. But what is so special about these particular trigonometric functions? The key to understanding why sines and cosines are so closely linked to repetitive events lies in thinking about motion in a circle.

Activity 1 *Making choices for learning*

An important aspect of learning is being able to make choices about what you intend to do based on aspects such as: your previous knowledge, your own strengths and areas you want to improve; feedback you have received; time-scales you are working to, and so on.

Take a few minutes to think about the work you intend to do to complete this unit. What are you basing your decisions on? You may find it helpful to think back to your work on earlier units. How did you decide what to concentrate on? How did you know when to 'stop' when working on a particular piece of work?

The Learning File Sheet for this activity invites you to keep a log indicating the work/activities you complete (or not) as you work through the unit, together with a justification for your choices.

1.1 Exercise bike

This subsection consolidates some ideas you met in *Unit 9* and *Unit 14*. Imagine you are sitting on an exercise bike. Your feet are pushing the pedals round and round at a steady rate. You are working hard—but not going anywhere!

► If you plotted the vertical height of one of your feet against time, what shape graph would you get?

Figure 1 shows the situation.

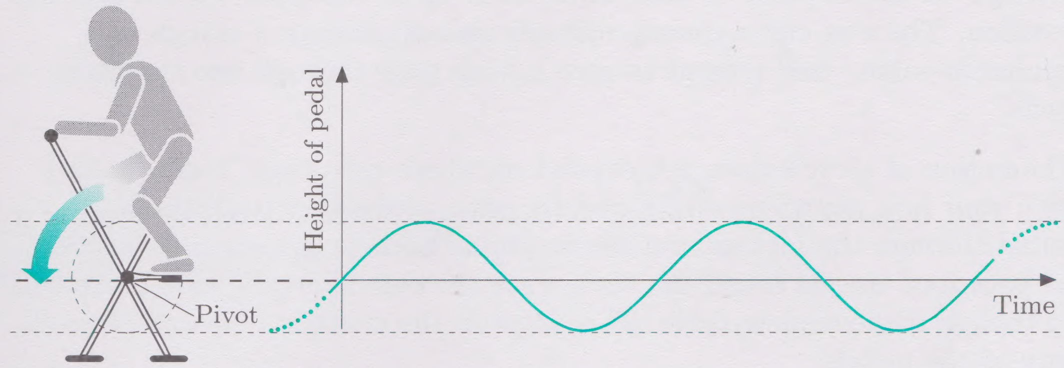


Figure 1 Pedalling an exercise bike

Assume you are pedalling so that the pedals go round at a steady speed of once a second. Your foot reaches its highest point when the pedal is at the top of its circular path, and the lowest when the pedal is at the bottom. As time passes your foot travels up and down, moving equally above and below the central pivot of the pedals each time round. If you plotted the height of your foot (relative to the central pivot) against time, you would get a curve whose shape you should recognize from earlier units.

The shape of the graph is the sine curve, but now it is linked to the circular motion of the pedals. As the pedals go round and round at a steady rate, the shape of the curve repeats itself producing the characteristic periodic, or regularly repeating, pattern of peaks and troughs.

For each complete revolution of the pedals, the sine curve passes through one complete cycle and then starts again. Look at Figure 2.

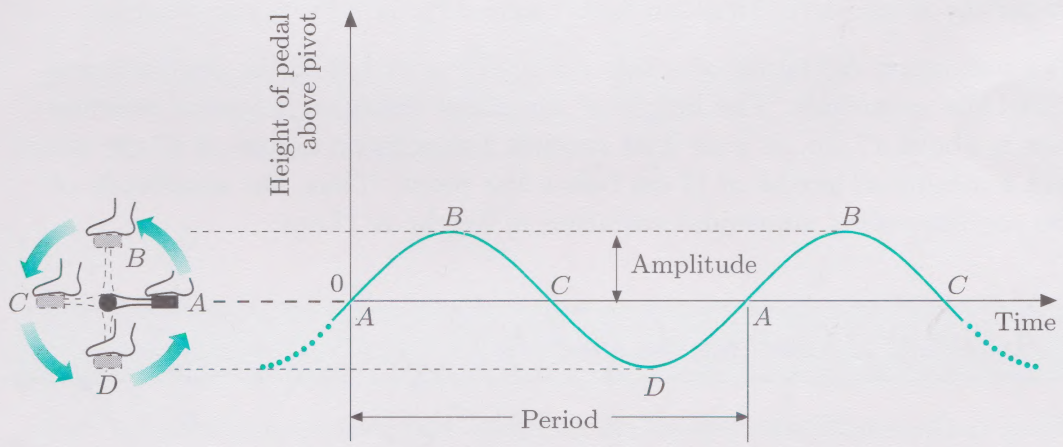


Figure 2 Period and amplitude of a sine curve

Suppose your foot is level with the pivot—at the three o'clock position A. If you measure height relative to this level, this height is zero. A quarter of a turn later your foot is at its highest point B, and the sine curve is at its peak. After another quarter turn your foot is level with the pivot again at

the nine o'clock position C, and the sine curve is back to zero. The bottom half of the turn mirrors the top half as your foot moves down, passes through its lowest point D and climbs back up to the three o'clock starting position. The sine curve correspondingly passes through a trough—its minimum value—and returns to zero having gone through one complete cycle.

The notion of a cycle does not depend on where you start. You can start with your foot anywhere on its circular path, and follow it all the way round through the highest and lowest points back to the starting position. As your foot travels round the circle once the corresponding sine curve will go though one complete cycle. The shape of the curve is repeated for each turn of the pedals.

Look at Figure 2 again. The time required for one complete cycle of a sine curve is called the *period*. If you are pushing the pedals round one complete turn every second, the sine curve will have a period of one second. If you are pedalling faster and manage two complete turns every second, then the sine wave will go through two complete cycles in one second. The period, the time for one cycle, will be half a second.

Recall inverse proportional relationships from *Unit 13*.

The number of cycles that a sine curve goes through in one second is called the *frequency*. Frequency and period are inversely proportional to each other. If it takes 3 seconds to complete one cycle (so the period is 3 seconds), then in one second the sine curve goes through $1/3$ of a cycle (the frequency is $1/3$ cycles per second). If the frequency is 5 cycles per second, then the period—the time to complete one cycle—is $1/5 = 0.2$ seconds.

In general, more cycles per second means a higher frequency and consequently a shorter period. Conversely, fewer cycles per second means a lower frequency and a longer period. Recall from *Unit 9* the unit of frequency is the hertz (written Hz), where 1 Hz is 1 cycle per second.

The maximum deviation of a sine curve above or below the centre line is called the *amplitude*. The length of the pedal crank on a typical exercise bike is about 17 cm, so your foot reaches a maximum height of 17 cm above and a minimum height of 17 cm below the pivot. Thus, the amplitude of the corresponding sinusoidal variation in height is 17 cm.

The adjective 'sinusoidal' means 'like a sine curve'.

Activity 2 Reading the sines

What is the amplitude, period (in seconds) and frequency (in Hz) of each of the three sine curves shown in Figure 3(a), (b) and (c).

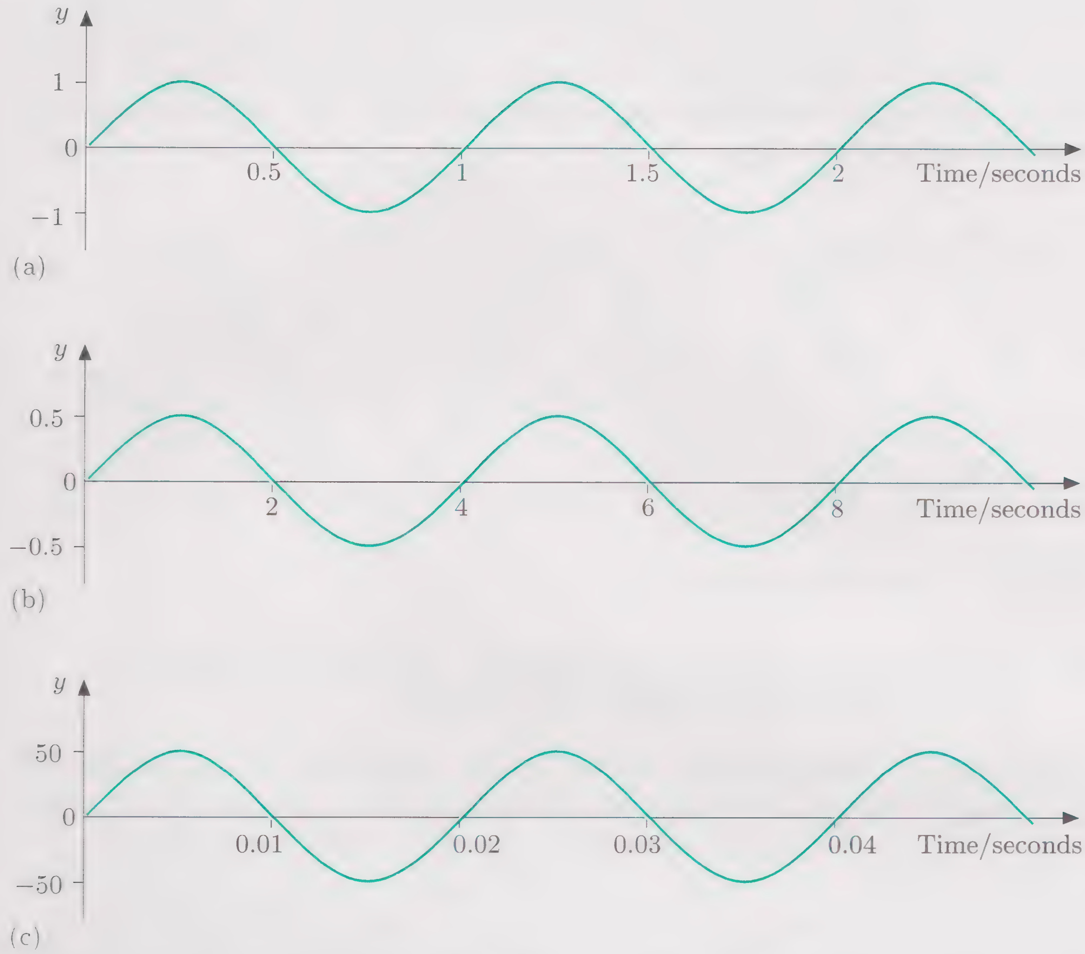


Figure 3 What are the amplitudes, periods and frequencies?

1.2 Sine writing

The circular motion of the pedals of the exercise bike produces or 'generates' the sine curve representing the height of the pedal relative to the pivot. Rotation and sine curves are intimately linked and share some mathematical language. When dealing with rotating things and sine curves the natural way to talk is in terms of angles. You know that one complete turn, or once around a circle, is equal to 360 degrees, but degrees are not the only way to measure angle. In *Units 9* and *14*, you saw that a different measure, sometimes called *radian* or *circular measure*, is also used. In circular measure, one complete turn around a circle is equal to 2π radians.

Activity 3 Pieces of pi

How many radians correspond to a quarter turn, a half turn and a three-quarter turn around a circle?

In Chapter 9 of the *Calculator Book*, you saw that if you plotted the shape of the function $y = \sin x$ on your calculator you would get a graph similar

to that in Figure 4. The plot reaches a maximum of 1 and a minimum of -1, so the amplitude of the sine curve is 1. Notice that the x-axis is scaled in radians where x is an angle. As x goes from zero to 2π radians, corresponding to one complete turn around a circle, the graph of $\sin x$ goes through one cycle.

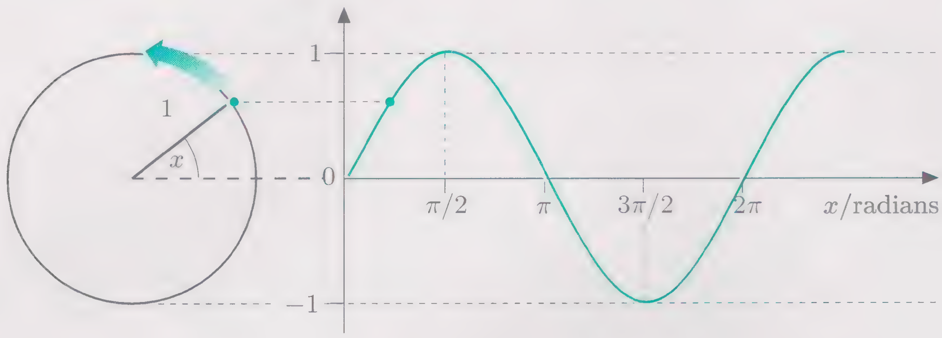


Figure 4 Generating the graph of $y = \sin x$

▶ How would you use the sine function to model the features of the curve produced in the exercise bike example?

Take stock of what you know so far. On the exercise bike, the variation of the height of the pedals with time is sinusoidal with an amplitude of 17 cm and a period of T seconds, the actual value of T depending on how fast you are pedalling. In contrast, the mathematical sine curve in Figure 4 has an amplitude of 1 and a period of 2π. So you must make two changes to the mathematical curve to make it match the actual curve; the amplitude must be scaled so that it matches the pedal amplitude, and the variable x must be modified so that the two periods match.

All you need to do to change the amplitude of the mathematical curve from 1 to 17 is to *scale* or multiply the sine function by 17. If y represents the variation in height of the exercise bike's pedals in centimetres, the changing height can be represented by the equation:

$$y = 17 \sin x$$

The next step is to link the variable x to the speed of rotation of the pedals. Look at Figure 5(a), which represents the variation in height of the pedals plotted against time when the pedals complete one rotation every second. The period of rotation is 1 second. Figure 5(b) shows the variation when the pedals are going round twice as fast. In this case, the period of rotation is 0.5 seconds. Figure 5(c) shows the general case for this particular exercise bike. On the horizontal axis, time is represented by the variable t, and T is used to represent the period—the time for one complete rotation of the pedals.

Figure 5(d) shows the curve of $17 \sin x$ plotted against x. As the sine curve goes through one cycle, the value of x goes from 0 to 2π radians. If you compare (c) and (d) you should be able to see a relationship between the way the variable t varies and the way the variable x varies.

▶ Can you see what it is?

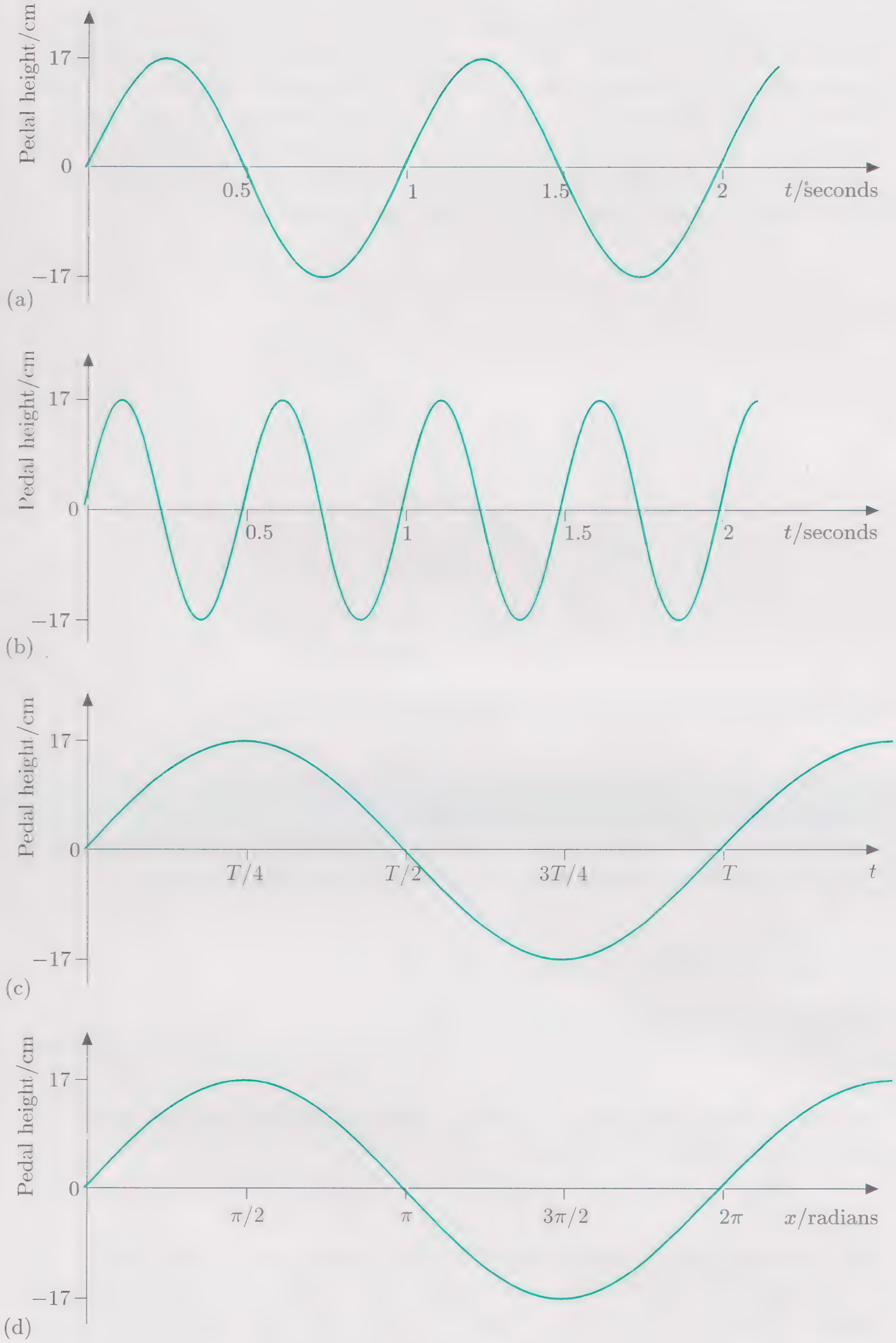


Figure 5 Sine curves of period (a) 1 second (b) 0.5 seconds (c) T seconds (d) the curve $y = 17\sin x$

As t grows from 0 to $T/4$ to $T/2$ to $3T/4$, and so on, the shape of the curve in (c) is the same as the shape of the sine curve in (d) as x grows from 0 to $\pi/2$ to π to $3\pi/2$. By comparing the shapes of the curves, you can see that $t = 0$ corresponds to $x = 0$; $t = T/4$ corresponds to $x = \pi/2$; $t = T/2$ corresponds to $x = \pi$; and so on. If you plot the corresponding values of x and t on a graph you will get a straight line as in Figure 6. The straight line passes through the origin of the graph. This means that the relationship between x and t is a directly proportional one.

Recall directly proportional relationships in *Unit 7* and *Unit 13*.

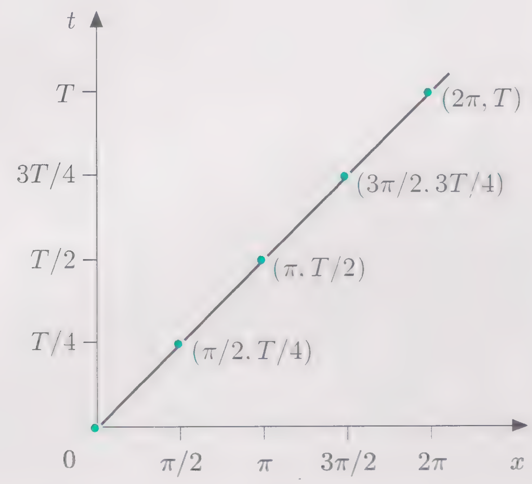


Figure 6 Directly proportional relationship between x and t

Activity 4 Finding the constant

The relationship between x and t can be written as the formula:

$$t = mx$$

Use Figure 6 to find an expression for m.

Rearrange the formula to make x its subject.

Rearranging formulas was discussed in *Unit 8*.

You should have found from Activity 4 that the directly proportional relationship between x and t is:

$$x = \frac{2\pi}{T}t$$

You can check this by substituting different values of t. At the start $t = 0$, so x is equal to $(2\pi/T) \times 0$ which is also zero. When the pedals have gone halfway round $t = T/2$, so $x = (2\pi/T) \times T/2 = \pi$. When the pedals have completed one turn $t = T$, and so $x = (2\pi/T) \times T = 2\pi$.

Recall that the sine curve model representing the height y of the pedals (relative to the pivot) is:

$$y = 17 \sin x$$

Now you can replace x by $\dfrac{2\pi}{T}t$ so that the model gives the height of the pedals as a function of the time t. That is, the function provides a mathematical model of the relationship between the height y (measured in centimetres) and the time t (measured in seconds) for any particular period of rotation T:

$$y = 17\sin\left(\frac{2\pi}{T}t\right)$$

Example 1

Suppose the pedals are going round with a period of 0.5 seconds.

(a) Sketch the variation of height with time.

(b) When do the pedals first reach their peak height?

(c) What is the height of the pedals after 0.3 seconds?

The period T is 0.5 seconds. So $2\pi/T = 2\pi/0.5 = 4\pi$, and the model is:

$$y = 17\sin(4\pi t)$$

Figure 7 shows the variation of height plotted against time. The peak height is first reached when $y = 17\sin(4\pi t) = 17\,\text{cm}$. This occurs after a quarter of a period when $t = T/4 = 0.5/4 = 0.125$ seconds. You can check this by substituting 0.125 for t in the formula:

$$y = 17\sin(4\pi \times 0.125) = 17\sin\frac{\pi}{2} = 17\,\text{cm}$$

After 0.3 seconds, the height of the pedals is given by:

$$y = 17\sin(4\pi \times 0.3) = 17\sin(1.2\pi) = 17 \times -0.588 = -9.99\,\text{cm}$$

This position is shown in Figure 7.

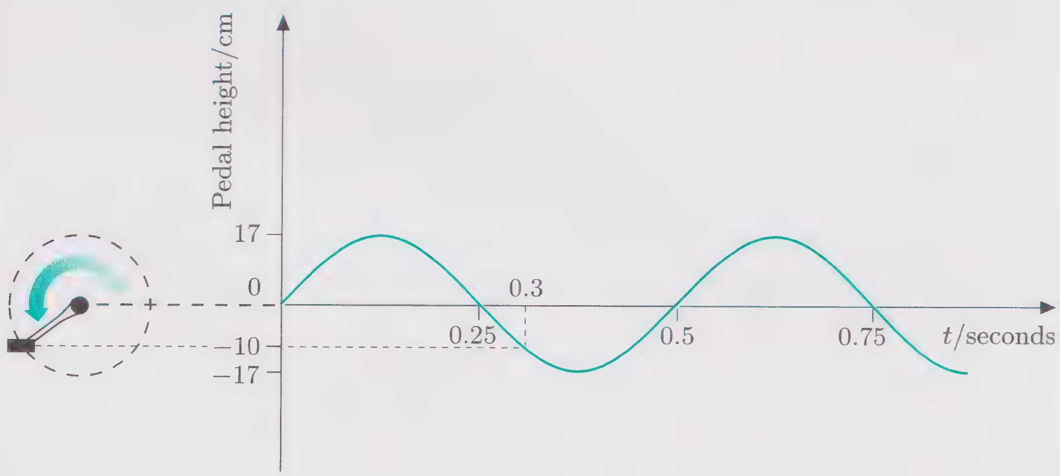


Figure 7 Height of pedals plotted against time

It is often easier to talk about the frequency of a periodic event—how often it occurs in a given time—rather than the period—the time for just one cycle to occur. In Subsection 1.1, you saw that frequency and period are inversely proportional to each other. More than this, one complete cycle per second corresponds to a frequency of 1 Hz, two cycles per second is 2 Hz, and so on. If f represents frequency in Hz and T represents the period in seconds, then:

$$f = \frac{1}{T} \quad \text{or} \quad T = \frac{1}{f}$$

This relationship can be used to express the formula for the sine curve in terms of frequency rather than period. Starting with:

$$y = 17\sin\left(\frac{2\pi}{T}t\right)$$

and replacing $\frac{1}{T}$ by f, the formula becomes:

$$y = 17\sin(2\pi f t)$$

Activity 5 Reading the sines

A sine curve is used to model a periodic variation with a frequency of 3 Hz and an amplitude of 0.1.

(a) What is the formula describing this curve?

(b) What is the period of the sine curve?

(c) Sketch, or use your calculator to display, the shape of this sine curve over the time interval $t = 0$ to $t = 1$ second to find when the first peak occurs.

Frequency is a measure of the number of cycles completed every second, and each complete cycle corresponds to 2π radians. Therefore, multiplying the frequency by 2π gives the number of radians turned through each second.

The angular frequency in radians per second is always a larger number than the corresponding frequency in hertz.

If f is the frequency in hertz, then the quantity $2\pi f$ is called the *angular frequency*. Angular frequency is measured in radians per second and is usually denoted by the symbol ω (the Greek letter 'omega'). For example, if a frequency f is equal to 1 Hz then the corresponding angular frequency ω is 2π, about 6.28 radians per second. Similarly, a frequency of 50 Hz corresponds to an angular frequency of $50 \times 2\pi = 100\pi$, some 314.2 radians per second.

Recall from *Unit 7* and *Unit 11* the distinction between speed and velocity.

Since a radian is a measure of angular distance, the number of radians turned per second is a measure of angular speed. When there is a direction involved—turning can be forwards or backwards—the term *angular velocity* is used. For example, working the exercise bike so that the pedals turn once a second in the forward direction means that their angular

velocity is 2π radians per second. An angular velocity of -2π radians per second would indicate that you were backpedalling at the same speed.

So there are two terms you might come across to describe the quantity $2\pi f$: angular velocity and angular frequency. Which one is used depends on the context. If you are dealing directly with rotating things where the velocity of rotation has a physical meaning (as with a wheel or a motor or a washing machine drum), then angular velocity is a natural idea to use. On the other hand, you may come across sine curves which are not the result of physical rotation (they may be modelling the sound of a musical note or an electrical signal, for example) or which are simply mathematical entities in their own right. In this case, the term 'angular frequency' is more likely to be used.

In science and technology as well as mathematics, the formula for a sine curve is often expressed using angular frequency rather than frequency in hertz. You should be prepared to handle both. Using angular frequency, the formula for the height of the pedals can be written in yet another way.

Replacing $2\pi f$ by ω in the formula $y = 17\sin(2\pi ft)$ gives the new formula:

$$y = 17\sin\omega t$$

There is a useful relationship between ω and the period T. Since ω is equal to $2\pi f$, the frequency f is $\omega/2\pi$. Now $T = 1/f$, so substituting for f gives:

$$T = 2\pi/\omega$$

Multiplying both sides of this equation by ω gives the relationship $\omega T = 2\pi$. In other words, multiplying the period by the angular frequency (where both are in appropriate units such as seconds and radians per second) always gives the answer 2π. This constant relationship is useful in working out T if you know ω, or working out ω if you know T. It is also useful as a check on calculations: the value of ωT must always be equal to 2π.

This also says that ω is inversely proportional to T.

Example 2 Middle C

A tuning fork sounding middle C (tuned to concert pitch) vibrates 261.6 times per second, producing a pure note at a frequency of 261.6 Hz. What is the corresponding angular frequency? What is the period T of the vibration?

The angular frequency $\omega = 2\pi \times 261.6 = 1643.8$ radians per second.

If you are working with frequencies in hertz, then the calculation of the period is straightforward:

$$T = \frac{1}{f} = \frac{1}{261.6} = 0.00382 \text{ seconds}$$

or just under four thousandths of a second.

If you are working with frequencies in radians per second, then you can use the following relationship:

$$T = \frac{2\pi}{\omega}$$

So here $T = 2\pi/1643.8 = 0.00382$ seconds.

As you would expect, the period comes out the same whichever way you carry out the calculation.

Recall a similar diagram in *Unit 7* to help remember the relationships between time, speed and distance.

Remembering the formulas

There is a useful way of remembering the relationships between ω, T and 2π, shown in Figure 8. Draw a circle containing a 'Y'-shape. Put 2π at the top of the 'Y' (remember **TwO Pi**) and ω and T underneath. Then to find the formula for angular frequency cover up 'ω' and you are left with $2\pi/T$. Similarly covering up 'T' gives the formula $2\pi/\omega$ for the period. And finally covering up '2π' gives the formula ωT.

The angular frequency ω and the period T are inversely proportional to each other (as were f and T), with 2π as the constant of proportionality.

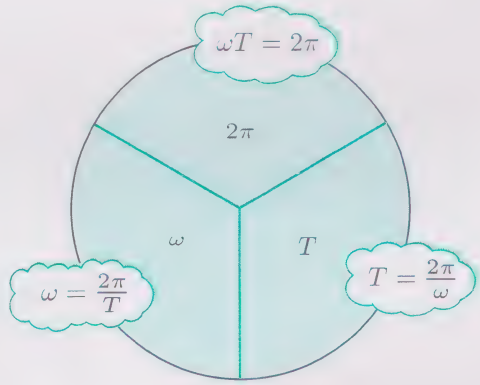


Figure 8 Relationship between ω, T and 2π

So there are alternative ways of writing the formula for a sine curve of amplitude A. If you know the period T, the variation of y with time t is described by the formula:

$$y = A\sin\left(\frac{2\pi}{T}t\right)$$

The period T (in seconds) is related to the frequency f (in hertz) by $f = \frac{1}{T}$, so the formula can also be written as:

$$y = A\sin(2\pi f t)$$

And if you are working with frequencies expressed in radians per second, $2\pi f$ can be replaced by the angular frequency ω, giving the formula:

$$y = A\sin\omega t$$

All these are different ways of saying the same thing. The main point to remember is that the values of $(2\pi/T)t$, $2\pi ft$ and ωt are all expressed in radians. When you come across a formula for a sine curve, you should be able to identify which part of the formula gives information about the amplitude, and which part gives information about the frequency or period.

Activity 6 *Interpreting the formula*

The vertical position y (expressed in metres) of a needle on a sewing machine, when it is working at a particular speed, is described by the formula:

$$y = 0.001 \sin 30t$$

(a) What is the amplitude and the period of the needle's motion?

(b) Display two complete cycles of the sine curve on your calculator, and check how long the two cycles take.

(c) At a different speed, the amplitude of the needle's movement is the same, but the period is 0.125 seconds. What is the formula for y, the needle's position?

Activity 7 *Handbook activity*

Add the important points from this section to your previous Handbook entries on the sine function.

In summary, a sine curve is a defined mathematical shape that can be used to model periodic behaviour. The curve is described by two parameters, the amplitude and the period (or frequency). The amplitude gives the 'height' of the curve—the maximum deviation of the peaks from the centre line. The period gives the 'width' of the curve—the time for one complete cycle (the frequency gives the number of cycles per second).

You saw how a sine curve is generated by steady circular motion. The amplitude is related to the radius of the circle and the period to the time taken for one complete rotation or turn around the circle. The number of cycles completed in one second is called the frequency. Frequency is measured in hertz, where 1 Hz is equal to 1 cycle per second.

In mathematics, angles and turning are described using radians. One complete rotation, cycle or period is associated with 2π radians. The angular frequency ω, measured in radians per second, gives the number of radians turned in one second. Angular frequency is related to the frequency in hertz by the formula $\omega = 2\pi f$.

Outcomes

After studying this section, you should be able to:

- ◇ use the following terms accurately and explain them to someone else: 'sine curve', 'amplitude', 'period', 'frequency', 'angular frequency', 'cycles per second', 'hertz', 'radian' (Activity 7);
- ◇ explain in your own words to someone not taking the course the relationship between circular motion and a sine curve (Activity 7);
- ◇ explain and use the mathematical relationships between frequency, angular frequency and period (Activities 4, 5 and 6);
- ◇ write down the formula for a sine curve given the amplitude and the frequency or period (Activities 2, 5 and 6).

2 Sunrise, sunset ...

Aims The main aim of this section is to give an example of the use of a sine curve in modelling. ◇

This section looks at how the annual repeating pattern of sunset times can be modelled using a mathematical formula. In the UK, it is part of everyday experience that the times at which the sun rises and sets each day depend on both the time of year and the place. In London, for example, the midwinter sun rises about 08.00 GMT (Greenwich Mean Time) and sets just after 15.50 GMT. In Glasgow, about 480 km to the north and 260 km to the west of London, the sun does not rise until after 08.45 GMT and sets just around 15.45 GMT, making Glasgow's day about 50 minutes shorter than London's. In midsummer, the situation is reversed; Glasgow's day starts at about 03.30 GMT, and is about 17.5 hours long. London's day starts about ten minutes later and is about an hour shorter. From midwinter to midsummer and back again, the sunrise and sunset times go through a pattern of values that repeats every year.

Some daily newspapers publish the times of sunrise and sunset, usually for one particular place, and some diaries contain the information as a table of places and times for each week of the coming year. Tables of data may provide a convenient way to store information, but lists of numbers provide little insight into any patterns or trends which may underlie the data. Plotting sunrise or sunset data on a graph gives a visual indication of how the times vary over a year, and is a useful first step to representing the data by a formula rather than a graph.

▶ From your own experience, what shape of curve do you think you would get if you plotted the time of sunset in a particular place over a year?

2.1 A sine of the times

In the UK, the time of sunset governs 'lighting-up time', when street lighting is switched on and when vehicles must by law turn on their lights. Figure 9, overleaf, shows a graph of the times of sunset in London plotted over an interval that includes all of the year 1996. Each data point represents the time of sunset on the same day (Saturday) of a particular week. Week zero represents the week from Saturday, 6 January 1996 to Friday, 12 January 1996, while week 51 represents the week from Saturday, 28 December 1996 to Friday, 3 January 1997.

Sunset times for specific days in a particular week can be estimated from the weekly data.

To give you a better feel for the overall shape of the graph, the horizontal scale has been extended back from week 0 to week −8 to include sunset data for the last eight weeks of 1995, and it continues beyond week 51 to include data for the first few weeks of 1997.

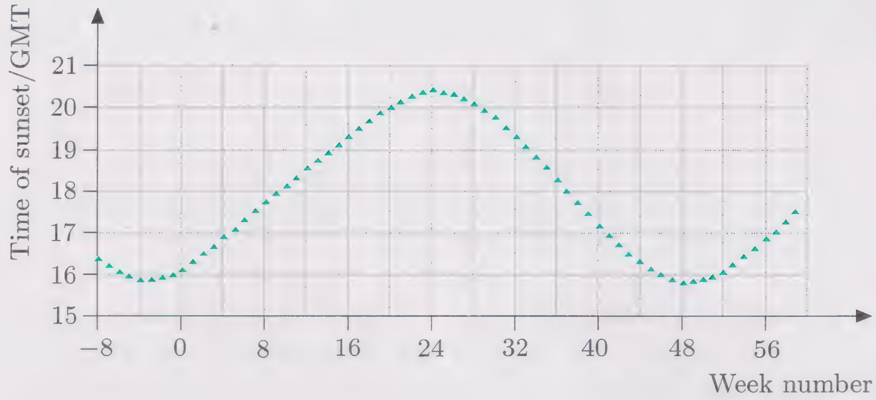


Figure 9 Variation in the time of sunset in London during 1996

The time of sunset varies smoothly from 15.52 GMT in winter to 20.22 GMT in the summer, a total range of 4.5 hours. From winter to summer and back to winter again, the curve goes through all its possible values. Governed by the steady motions of the Earth, this cycle repeats (with very small variations) year after year after year. It is another example of periodic behaviour.

To predict the time of sunset for any day during a year, astronomers use a mathematical model which takes into account the exact position of the Earth in its orbit around the sun. But you are not going to see how to reconstruct that formula. The aim of this section is to set up a model of the periodic variation of sunset times which uses only the data in Figure 9 and the mathematics you have met so far.

Mathematical models are used for a variety of reasons, and come in a variety of forms. If you simply want numerical information about an event in the future, then a list of data gives the information in a convenient form. If you want to see whether there are any patterns or regularities in the data, then a graph may be the most appropriate way to store and present the information. But if you want to transform the model using algebraic manipulation, you need to be able to set up a formula.

In Block C, you used your calculator to find formulas to model collections of data. You saw how you could fit straight lines, parabolas, exponential curves and power laws to data using regression techniques. In doing so, you were moving from graphical to algebraic mathematical models, using the characteristic shape of a particular mathematical curve to represent a collection of data points by a single formula.

▶ What kind of mathematical formula would you choose to describe the annual pattern of sunset times?

The variation in the time of sunset over a year looks promisingly like a sine curve—although it is not a perfect sine curve. The peak of the curve is around week 24, coming about 27 weeks after the preceding midwinter trough and about 25 weeks before the following trough. Unlike a sine curve, therefore, the plot of sunset times is not quite symmetrical. Nevertheless, identifying the general shape is a good start. In any

modelling activity, it is useful to start with the shapes and the mathematics that you already know about.

Looking at the general shape can also tell you which curves will probably not qualify as models, at least over the range you are interested in. Clearly a straight line will not do—but what about other functions? The sunset curve is smooth and periodic—it repeats almost exactly year after year. Curves such as parabolas or exponentials or power laws may be made to fit different parts of the periodic curve with varying degrees of accuracy, but none has the property of smoothly repeating the same shape over and over again. The only functions you have met which do this are the trigonometric ones: sine and cosine.

The tangent, cotangent, secant and cosecant trigonometric functions you met in *Unit 14* are periodic, but they do not repeat smoothly. Use your calculator to plot them and see for yourself where the 'breaks' come.

So choose a sine curve as the basic model and see how far it takes you. In making this choice, you are stressing the features that the sunset curve has in common with a sine curve—its periodicity, its constant amplitude, its smooth variation from minimum to maximum and back to minimum over a cycle, ignoring the fact that the sunset curve is not a perfect sine curve.

Recall that the basic sine curve model has the formula:

$$y = A\sin\left(\frac{2\pi}{T}t\right)$$

where A is the amplitude of the sine curve and T is the period.

Activity 8 Choosing the amplitude and period

Use the graph of sunset times in Figure 9 to choose suitable value for the amplitude A and the period T of the sine curve model.

If y represents the time of sunset in GMT and t is the week number, then a first stab at writing down a model might lead to something like this:

Note that time should be expressed as a decimal.

$$\text{time of sunset} = 2.25\sin\left(\frac{2\pi}{52}t\right)$$

Figure 10 shows the sunset times predicted by this model plotted against the week number.

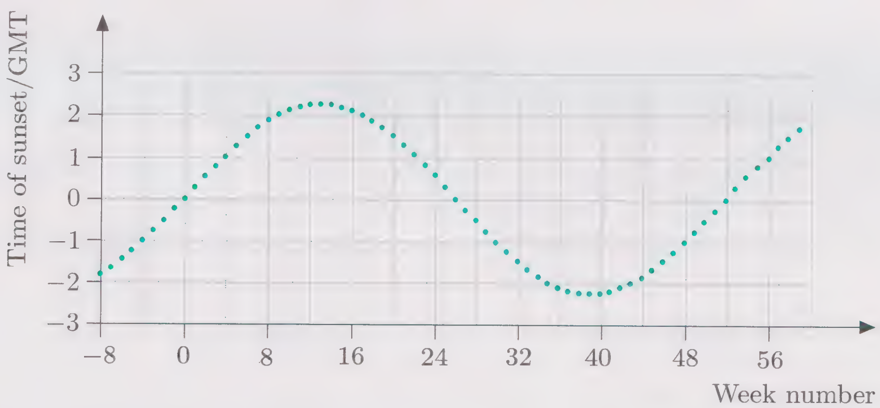


Figure 10 Plot of $2.25\sin\left(\frac{2\pi}{52}t\right)$

The resulting curve has the right amplitude of 2.25 hours and the right period of 52 weeks, but if you compare it with Figure 9 you can see something is wrong: the curve has peaks in the wrong places and the time of sunset has negative as well as positive values.

These problems are fairly easily remedied, however.

▶ What must be done?

First, the curve must be shifted up so that it no longer varies up and down about zero. Over the year, the time of sunset in London varies from 15.52 GMT in midwinter to 20.22 GMT in midsummer. Assuming a symmetrical variation, the mean or average time of sunset is midway between these two times, at 18.07 GMT. Expressed as a decimal, 18 hours 7 minutes is 18.12 hours.

Recall the discussion on translating curves horizontally and vertically in *Units 10, 11* and *13.*

So the curve must be shifted up so that it varies above and below the value 18.12, rather than zero. To shift or *translate* the curve vertically, simply add the required amount to the formula:

$$\text{time of sunset} = 18.12 + 2.25 \sin\left(\frac{2\pi}{52}t\right)$$

Figure 11 shows the predictions of sunset time from the revised model plotted for comparison with the actual data.

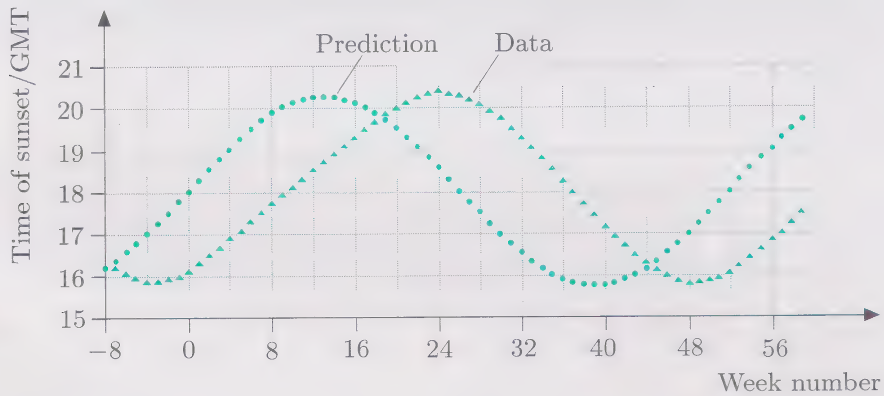


Figure 11 Sunset data and the predictions of the revised model

Activity 9 Displaying the predictions

Enter the sunset time formula in your calculator and display the graph. Check for yourself that it has the shape shown in Figure 11.

The model is looking better, but the peaks are still in the wrong places.

▶ What should be done so that the model is a better fit to the data?

To match the model to the data, the sine curve must be shifted or translated to the right. But by how much? Before tackling this question look at Figure 12, which shows two sine curves. The black curve has the formula $y = \sin x$. At $x = 0$, the curve passes through zero, reaches its peak of $+1$ at $x = \pi/2$, returns to zero at $x = \pi$, and so on. The coloured curve has the formula $y = \sin(x - \pi/2)$. It is the same in all respects except it is translated by a quarter of a period—by $\pi/2$ radians—to the right relative to the first. So the second curve passes through -1 at $x = 0$, zero at $x = \pi/2$, $+1$ at $x = \pi$, and so on.

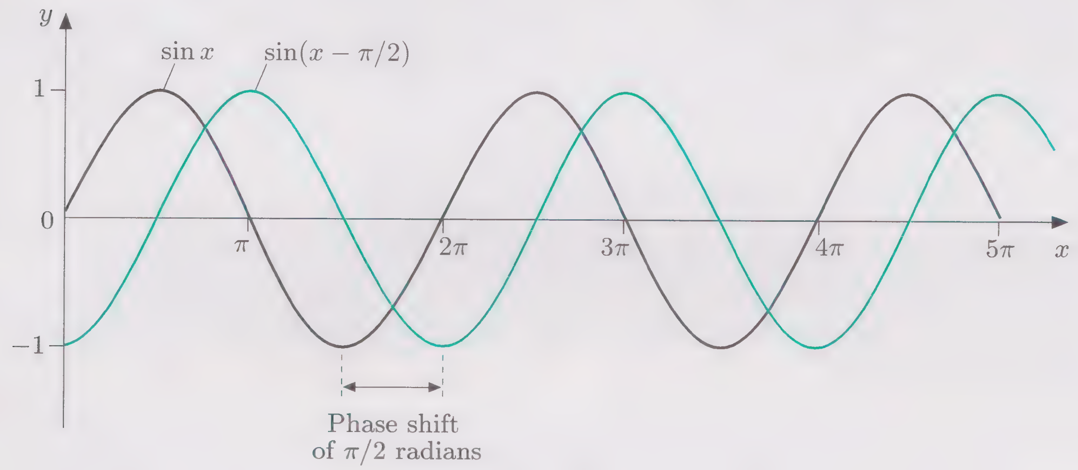


Figure 12 Phase shift between two sine curves

The shift between the two sine curves in Figure 12 is called a *phase shift*: $\sin(x - \pi/2)$ has a phase shift of $\pi/2$ radians relative to $\sin x$. The curve of $\sin(x - \pi/2)$ is said to 'lag' that of $\sin x$ by $\pi/2$ since it appears to follow the same values of $\sin x$ at a later value of x. The peak of $\sin(x - \pi/2)$, for example, occurs at $x = \pi$, a quarter of a period after the peak of $\sin x$ at $x = \pi/2$. You can also look at this the other way round and say that the curve of $\sin x$ comes before the curve of $\sin(x - \pi/2)$. Sin x is then said to 'lead' $\sin(x - \pi/2)$.

Equivalently, you can talk of a phase shift of 90 degrees instead of $\pi/2$ radians.

Activity 10 Showing phase shifts

(a) Use your calculator to display the curve $y = \sin(x + \pi/2)$. For which values of x between 0 and 2π is the curve equal to: 0, -1, $+1$?

(b) Now display both $\sin x$ and $\sin(x + \pi/2)$. What is the difference between the two curves? Which curve leads and which lags?

Shifting a sine curve a quarter period to the right involves subtracting $\pi/2$ from x. Shifting a quarter period to the left, on the other hand, involves adding $\pi/2$ to x. You are not restricted to quarter period shifts however—a sine curve can be shifted left or right by any fraction of the period by adding or subtracting the appropriate amount. Adding $2\pi/3$

Shifting a sine curve horizontally by a fixed amount by adding to or subtracting from x is another example of translation.

radians, for example, shifts a sine curve to the left by $2\pi/3$ or one-third of the period. Subtracting 1 radian shifts or translates a sine curve to the right by $1/2\pi$, or 0.16 of the period.

Now return to the problem of adjusting the sunset time model. Look at Figure 11 again and notice that the nearly-sinusoidal curve of the actual sunset data lags the sine curve produced by the model. The model predicts that the time of sunset during week 0, represented by $t = 0$, is 18.12 hours (18.07 GMT). However, the data show that the sun does not set at this time until 10 weeks later. For the model to match the data, the sine curve must be shifted to the right by an amount corresponding to 10 weeks. Since the period of the data is 52 weeks, the shift must be $10/52$ of a period, corresponding to $(10/52) \times 2\pi = 1.21$ radians. Incorporating this shift in the model produces the modified formula:

$$\text{time of sunset} = 18.12 + 2.25 \sin\left(\frac{2\pi}{52}t - 1.21\right)$$

This formula is a mathematical model of the way the time of sunset changes over a year. But is the model any good? How well does it fit the actual data? To check you must compare the prediction of the model with the actual data.

In Figure 13, the predictions of the model are plotted on the same graph as the sunset data to show where the values differ. The sine curve of the model follows a slightly different path from the actual curve and you should be able to see where the two curves deviate from each other. The curves start together around week -3, corresponding to mid-December. Until week 10, the sine curve model underestimates the actual data, predicting sunset times that are slightly too early for this part of the year. The curves then cross over just before the Spring equinox and the sine model overestimates the data, giving sunset times which are too late, until week 23 (just before midsummer) where the model's peak occurs just before the actual peak. For the second half of the year, the model consistently gives sunset times which are slightly too early. The model and data coincide again around midwinter and the cycle then starts again.

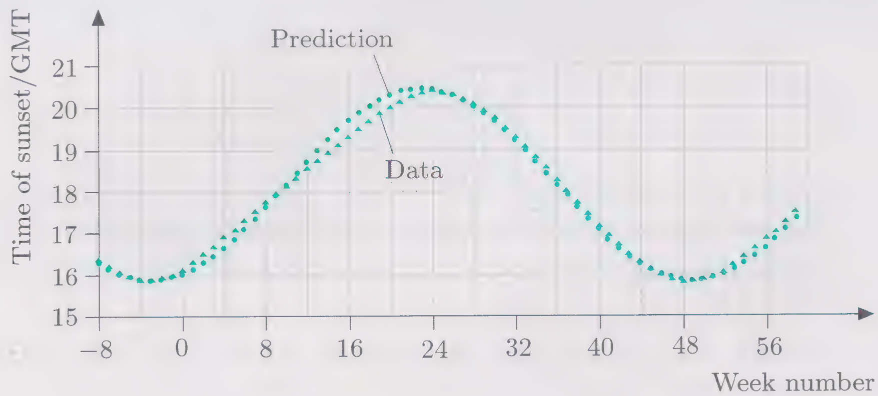


Figure 13 Plot of the sunset data and the predictions of the modified model

The expression $\left(\frac{2\pi}{52}t - 1.21\right)$ in the model looks rather daunting but in fact it reduces to nothing more than a number expressed in radians. Remember that as t goes from 0 to 52 (from the beginning of one year to the beginning of the next) in steps of 1, the expression $\frac{2\pi}{52}t$ goes from 0 to 2π in steps of $\frac{2\pi}{52}$, so the value of the whole expression goes from $(0 - 1.21) = -1.21$ radians to $(2\pi - 1.21) = 5.07$ radians, an overall change of 2π radians.

Example 3 May Day

The first day of May is the traditional beginning of summer in England. What time does the sun set at the start of the week in which May Day falls?

May Day occurs in week 16. So putting $t = 16$ in the formula gives:

$$\begin{aligned}
\text{sunset time} &= 18.12 + 2.25\sin\left(\frac{2\pi}{52}t - 1.21\right)\\
&= 18.12 + 2.25\sin\left(\frac{2\pi}{52} \times 16 - 1.21\right)\\
&= 18.12 + 2.25\sin(1.93 - 1.21)\\
&= 18.12 + 2.25\sin(0.72)\\
&= 18.12 + 2.25 \times 0.66\\
&= 19.61, \text{ or } 19.37\,\text{GMT}
\end{aligned}$$

The actual time of sunset for the start of this week is 19.18 GMT; the model overestimates the time by 19 minutes. As you will see shortly, it turns out that the predictions of the model are most inaccurate around this time of the year.

Activity 11 Remember, remember ...

On 5 November in the UK bonfires and fireworks celebrate Guy Fawkes' 'Gunpowder Plot', an attempt to blow up King James I and Parliament in 1605. Week 43 includes 5 November. Use the model to predict when the sun sets in London at the start of the week in which Guy Fawkes' Night falls.

An alternative way to assess how well the model's predictions fit the actual data is shown in Figure 14. This is a plot of the difference in minutes between the actual and the predicted values of sunset time over the year.

A positive difference indicates where the model overestimates the sunset time—the predicted time is too late—and a negative difference indicates where the model underestimates the data giving a sunset time that is too early.

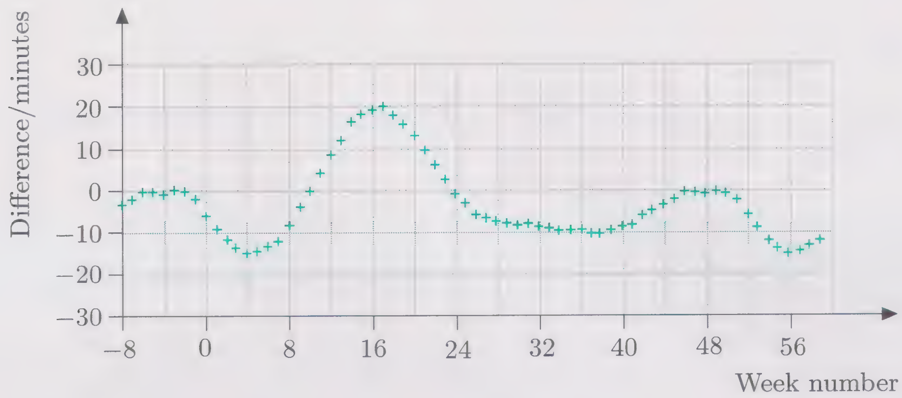


Figure 14 The difference between the model's predictions and the actual data

The advantage of this sort of plot is that it is easier to see where the match between predicted and actual data is good, and where it is less so. You can see that the difference, or error, around week −4 is close to zero indicating a good match. Around week 4 the error reaches −15 minutes before passing through zero at week 10 to reach a maximum of +20 minutes around week 17; here, the match between model and data is poorest. The error returns to zero around week 24 and then is no worse than about −10 minutes for the rest of the year.

Numerical accuracy is an important feature in many mathematical models. In this case, the greatest difference between the prediction of the model and the actual data is nowhere more than 20 minutes for a maximum variation of 4.5 hours. In other words, in the worst case the sine curve model predicts the time of sunset with an error of less than 8 percent. For a lot of the time the match is better than this: for 32 consecutive weeks the error does not exceed about 10 minutes, or 4 percent. It is up to the user of a mathematical model to judge whether the predictions offer an acceptable fit to the data. Decisions depend on what the model is for and on the criteria for accuracy.

The general shape of the curve produced by the model seems to be not too different from that of the actual data, and offers a reasonable visual image of how things change. Even though the curve is not an exact fit, the model offers a way of thinking about the actual data—the plot of the sunset data is *like* a sine curve. Similes like this often turn into metaphors; allowing talk about the data as if they were a sine curve, and to ascribe to the data mathematical characteristics such as amplitude, period and phase shift which strictly belong only to the model.

The differences between the curves may also prompt questions about *why* they are different. Does the model ignore some feature that should have been taken into account? Is there a good reason why the pattern of sunset times is not a perfect sine wave? Thinking about questions like these can

give insight not only into how the model may be improved but also into the mechanism of the physical phenomenon itself.

In this case, you were not looking for a model specifically for accurate prediction (recall that you started with available accurate data), but to find out what sort of match was possible with a model of this type. However, if you were looking for greater numerical accuracy, less than 4 percent error over the whole year say, then the next stage would be to think about how the model might be changed or modified to achieve the desired result. Any model will impose some limits to accuracy. It may be that modifying the parameters of the model slightly reduces the error or spreads it more evenly over the year. Or it may turn out that a better result simply cannot be achieved with a model of this type. It may be obvious which changes to make or it may not, there are no hard and fast rules. Producing an acceptable mathematical model in a particular situation requires a combination of mathematical skill, intuition and experience. Modelling is an art. The steps leading to the final model were:

Recall that *parameters* are the numbers in the model representing quantities such as amplitude and phase shift.

- ◇ the data points were plotted and showed a periodic variation;
- ◇ a sine curve was used to characterize the shape of the plot;
- ◇ the amplitude and the period were estimated from the plot;
- ◇ the sine curve was translated vertically by adding a constant to match the mean value of the sunset time;
- ◇ the sine curve was translated horizontally by including a phase shift to match the position of the plot along the time axis;
- ◇ at each stage, the match between the model and the sunset data was checked, and a decision was made about what to do next.

The modelling cycles you met in *Units 5 and 10* offer frameworks for thinking about modelling, but they do not guarantee results nor supply ideas about how to improve a model.

Activity 12 *Thinking about modelling*

Take some time now to think about the modelling process you have just been through. What assumptions have been made? Can you suggest any ways in which the model be improved to fit the data better? How would you define 'a better fit'?

Make some notes on your Learning File sheet.

Now work through Section 15.1 of Chapter 15 in the Calculator Book.

2.2 *Identity parade*

So far you have been using only the sine function to model periodic behaviour. But this is not the only function available to mathematicians. Very closely related is the cosine function, written $\cos x$. As the value of x increases from 0 to 2π, the value of $\cos x$ varies from $+1$ to -1 and back again. Plotting the value of $\cos x$ against the value of x, gives the cosine curve.

Recall the cosine function from *Units 9* and *14*.

Activity 13 Relating sines and cosines

Display the sine curve: the graph of the function $y = \sin x$; and the cosine curve: the graph of the function $y = \cos x$ on your calculator.

How are the curves related to each other?

You can think of the cosine curve as a sine curve that has been shifted to the left along the x-axis by an amount equal to $\pi/2$ radians, as in Figure 15(a). The relationship between the cosine and sine curves can be expressed mathematically by the equation:

You saw in *Unit 14* that there are special relationships among the trigonometric functions.

$$\cos x = \sin\left(x + \frac{\pi}{2}\right)$$

▶ How is this expression to be interpreted?

All it is saying is that the curves for the cosine and the shifted sine functions are always the same. That is, for any value of x, the corresponding values of $\cos x$ and $\sin(x + \pi/2)$ are the same. For example, if $x = 0.5$, then $\cos 0.5 = 0.8776$, and $\sin(0.5 + \pi/2) = \sin(2.0708) = 0.8776$. Try some other values for yourself. This relationship between $\cos x$ and $\sin x$ is an example of an *identity*, because it is true for all values of x. That is, for any value of x, the values of the cosine and the shifted sine functions are always the same.

You can alternatively think of a sine curve as a cosine curve that has been shifted to the right along the x-axis by $\pi/2$ radians, as in Figure 15(b). Recall from Subsection 2.1 that shifting to the right means subtracting an angle. Hence the relationship between the sine and shifted cosine curves is:

$$\sin x = \cos\left(x - \frac{\pi}{2}\right)$$

This is another identity, an expression that is true for all values of x.

Adding $\pi/2$ to the angle in a sine or a cosine curve moves it to the left by a quarter of a period, subtracting $\pi/2$ moves the curve a quarter period to the right. One way to remember which way a curve is shifted is to make up a mnemonic (a memory aid) to help you. Perhaps something like '**R**emoving $\pi/2$ means moving to the **R**ight', or '**R**emoving is a **R**ight move'.

Activity 14 From cosine to sine

Display the curve $y = \cos(x + \pi/2)$ on your calculator. How would you express the curve in terms of $\sin x$?

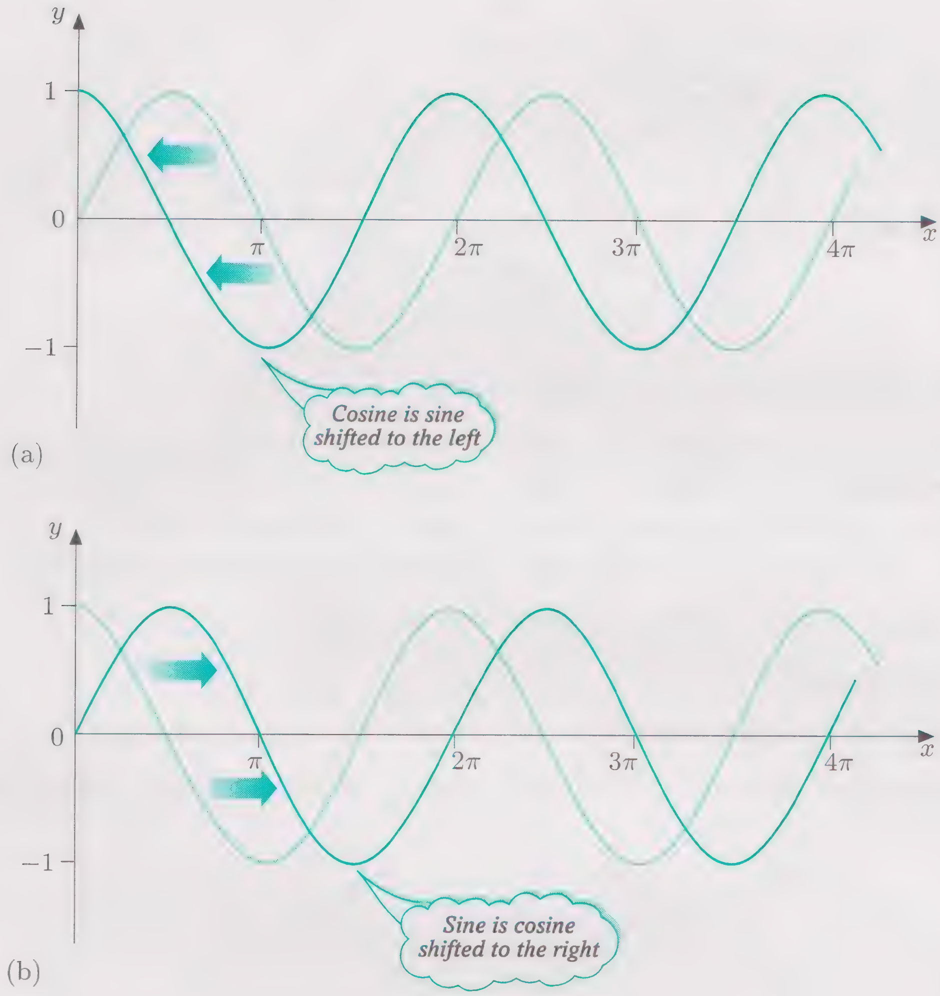


Figure 15

Shifting $\sin x$ or $\cos x$ by even multiples of π, that is $2\pi, 4\pi, 6\pi, \ldots$, simply gives you the same function again. This is because sine and cosine are periodic functions with a period of 2π. Look at Figure 16, overleaf. Moving through one or more complete periods (in either direction) takes you from your starting point to an identical y-value on the curve. This means that $\sin(x + 2\pi)$ is equal to $\sin x$, as is $\sin(x + 4\pi)$ and $\sin(x + 6\pi)$ for any value of x. Writing this relationship as an identity looks like this:

$$\begin{aligned}\sin x &= \sin(x + 2\pi)\\ &= \sin(x + 4\pi)\\ &= \sin(x + 6\pi)\end{aligned}$$

or more generally:

$$\sin x = \sin(x + n\pi)$$

which is true whenever n is an even number, for $n = 0$, 2, 4, 6, 8, and so on.

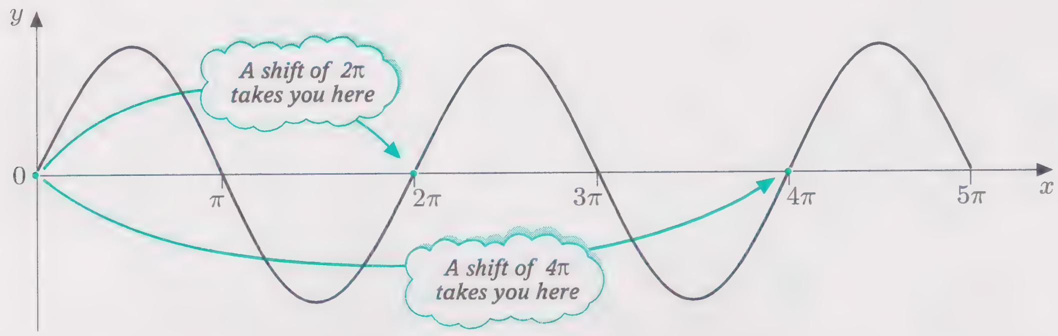


Figure 16 Shifting by even multiples of π

Shifting by odd multiples of π, that is $\pi, 3\pi, 5\pi, \ldots$ effectively turns the curve upside down. Figure 17 shows that the peaks and troughs of $\sin(x+\pi)$ are the opposite of those of $\sin x$. The same is true of $\sin(x+3\pi)$, $\sin(x+5\pi)$, and so on. So there are some more identities:

$$\sin x = -\sin(x+\pi) = -\sin(x+3\pi) = -\sin(x+5\pi)$$

or more generally:

$$\sin x = -\sin(x+n\pi)$$

which is true whenever n is an odd number, for $n = 1$, 3, 5, 7, and so on.

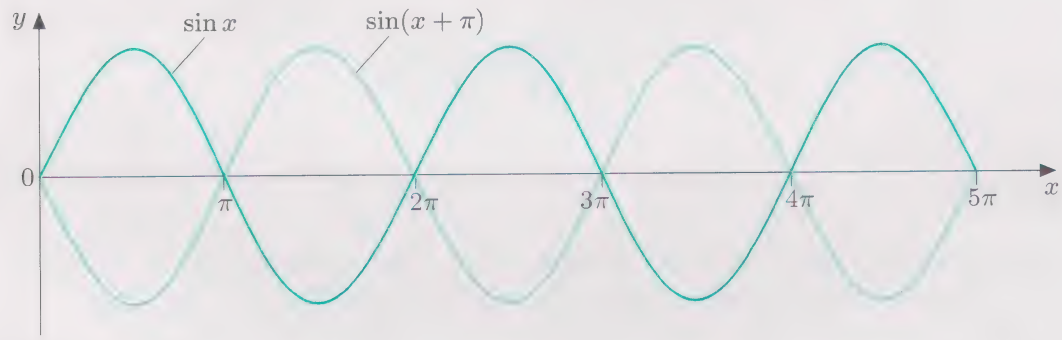


Figure 17 Shifting by odd multiples of π

These identities are also true for cosines because sines and cosines are just shifted versions of each other. If n is an even number:

$$\cos x = \cos(x+n\pi)$$

If n is an odd number:

$$\cos x = -\cos(x+n\pi)$$

Try out these identities on your calculator, and confirm for yourself the statement about odd and even values of n. Also try out the cases where n is negative. Sine and cosine identities are true for negative as well as positive values of n.

It is best not to try to remember all these relationships but rather to work them out as necessary, by drawing a sketch or using your calculator and thinking about what you must do to one side of the identity to make it equal to the other.

Activity 15 Display of identity

Use your calculator to display the following functions, and hence complete the table of sine and cosine identities. Comment on the shift you see.

Function	Identity	Comment
$\sin(x - \pi/2)$		
$\cos(x + 2\pi)$		
$\sin(x - \pi)$		
$\cos(x + 3\pi/2)$		

You may not have the same answers for the last activity—but that does not necessarily mean you are wrong. Because sine and cosine differ only by a $\pi/2$ phase shift—and because the functions are periodic—there are any number of correct ways of expressing these identities. Sometimes the reason for expressing a sine or a cosine function in a different form is to be able to write it more simply. Sometimes it is because it is easier to work with just sines or just cosines, and not have a mixture of both.

The important point here, however, is not that you worry about all the different ways sines and cosines can be expressed, but that you get some practice in displaying the curves on your calculator and get used to recognizing the effects of phase shifts on the relative positions of the peaks and troughs. Finally, remember that in an identity the curves described by the expressions on each side of the equals sign are identical. Your calculator will draw one curve on top of the other, so if you have an identity you will see only one curve.

Activity 16 From sine to cosine

Recall that the formula for the sunset time is:

$$18.12 + 2.25\sin\left(\frac{2\pi}{52}t - 1.21\right)$$

Write this model using a cosine instead of a sine function.

So far you have been dealing with sines and cosines as models of periodic behaviour, but you saw in *Unit 14* that they also occur as trigonometric functions, relating sides and angles in right-angled triangles. One important relationship between sines and cosines turns out to be another way of expressing Pythagoras' theorem.

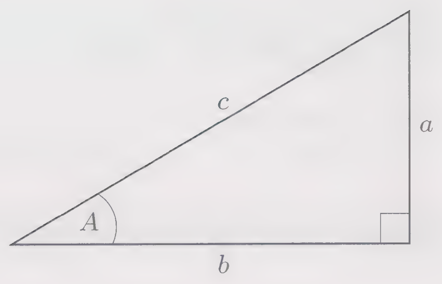


Figure 18 A right-angled triangle

Look at Figure 18. Pythagoras' theorem says that:

$$a^2 + b^2 = c^2$$

Dividing both sides by c^2 gives:

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

but $a/c = \sin A$ and $b/c = \cos A$ so the relationship is:

$$(\sin A)^2 + (\cos A)^2 = 1$$

This notational style is used for all trigonometric ratios.

A point of notation here: rather than putting in the brackets, it is usual to write $\sin^2 A$ instead of $(\sin A)^2$ and $\cos^2 A$ instead of $(\cos A)^2$. This formula is true for any angle, so using x to stand for a general angle gives the identity:

$$\sin^2 x + \cos^2 x = 1$$

The angle x in the triangle in Figure 18 was acute (less than a right angle). However, the identity holds for all values of x. This can be seen by defining the trigonometric function in terms of the coordinates of a point moving round a circle. This is a broader definition which extends the triangle definition and links the view of trigonometric function which you met in *Unit 9* to the perspective you met in *Unit 14*.

Look at Figure 19.

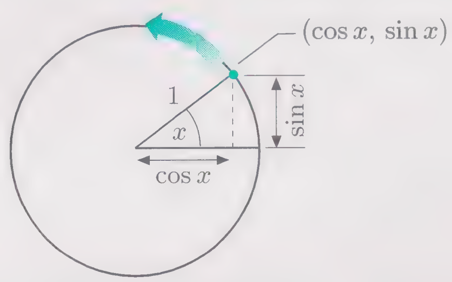


Figure 19 Defining the point $(\cos x, \sin x)$

The point (the pedal of the exercise bike) moves round a circle of radius 1 unit. The angle between the horizontal and the radius to the point is called x. Figure 19 shows x as an acute angle. From the triangle definition

of sine and cosine, you get:

$$\sin x = \frac{\text{opposite}}{1} \quad \text{and} \quad \cos x = \frac{\text{adjacent}}{1}$$

So the opposite side (the height of the point) is $\sin x$; this is the vertical coordinate of the point (relative to the centre of the circle). Similarly, the adjacent side is of length $\cos x$; this is the horizontal coordinate of the point. The coordinates of the point are $(\cos x, \sin x)$. This is true for all angles.

In Figure 20, the coordinates of any point on the circumference of a circle of radius 1 are $(\cos x, \sin x)$, where x is the angle measured *anticlockwise* from the horizontal axis. As the value of x goes from 0 to 2π, so $\cos x$ and $\sin x$ go through all their values (one complete period) and the point moves one complete turn around the circle. As the point goes around the circle, the vertical coordinate plotted against x traces out a sine curve, and the horizontal coordinate plotted against x traces out a cosine curve. So $\sin x$ and $\cos x$ are defined as the the vertical and horizontal components of a point on the circle.

Measuring angles anticlockwise from the horizontal axis is the convention used by mathematicians. Notice that it is different from that used by walkers and map users, who measure angles clockwise from the vertical axis.

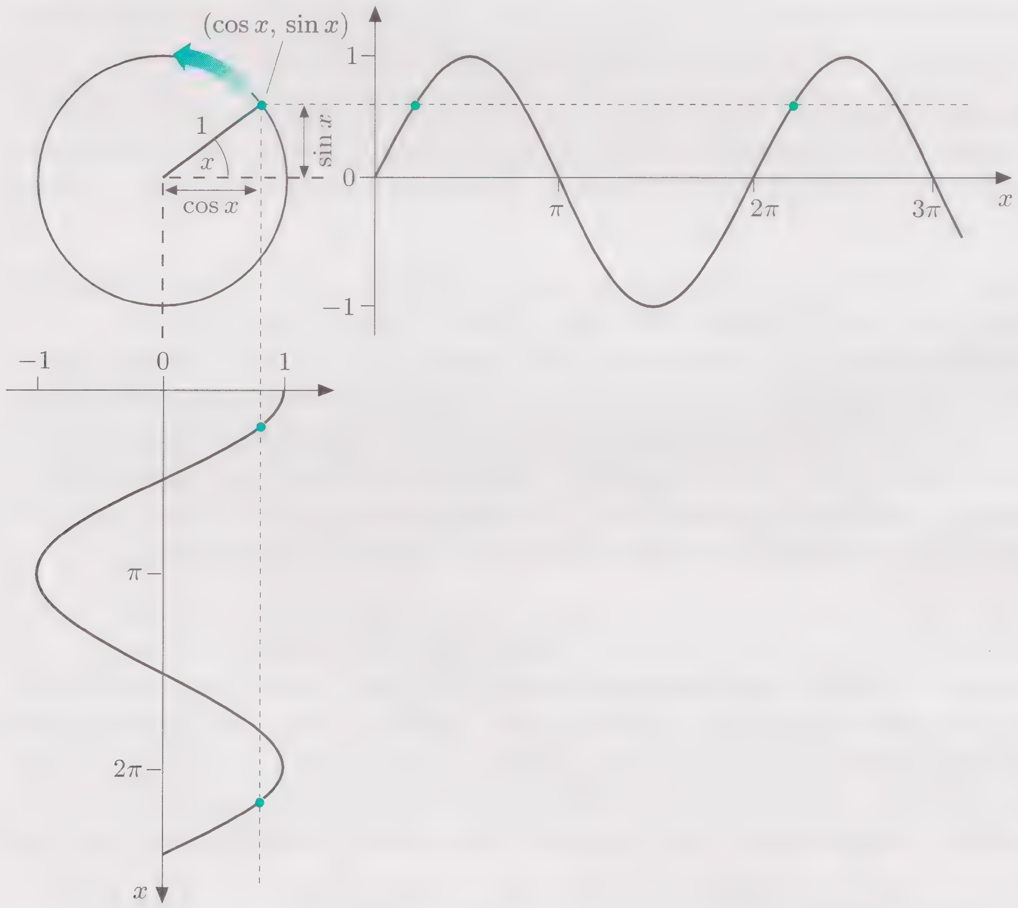


Figure 20 Sine and cosine curves related to circular motion

Linking the variations of both the horizontal and vertical coordinates of a point to one variable, in this case x, is the basis of a technique called *parametric graphing*. You can learn more about parametric plots in Section 15.2 of Chapter 15 of the *Calculator Book*.

Activity 17 *Identity check*

Use your calculator to convince yourself that the identity:

$$\sin^2 x + \cos^2 x = 1$$

is true for any value of x by displaying together the functions $y_1 = \sin^2 x$, $y_2 = \cos^2 x$, and their sum. Try different ranges of x. What do you see?

This section has looked at using the mathematics of sine and cosine curves to set up a mathematical model of variation of the time of sunset over a year. The general form of the model is:

$$y = M + A\sin\left(\frac{2\pi}{T}t + \phi\right)$$

Here y represents the time of sunset, M is the mean or average value of the sunset time, A is the amplitude of the sinusoidal variation, T is the period and ϕ (the Greek letter 'phi') is the phase shift (positive or negative) associated with the sine curve. The values of these parameters were chosen so that the sine curve matched the available data.

Sine and cosine curves are very closely related. Formulas using sines and cosines can be expressed in different ways using the idea of mathematical identities. A central idea is that $\sin x$ and $\cos x$ curves are identical except for a phase shift of a quarter period, or $\pi/2$ radians.

Graphs and formulas can both be used to model events in the real world. Graphs are useful because they give a visual representation of how quantities change and they store information in the form of points and lines. You can learn to interpret the resulting shapes by sight, and make judgements based on the shape the data makes. Mathematical formulas are useful because they can be used for detailed calculation and manipulated according to the rules of algebra. Formulas are a concise way of conveying information, replacing graphical shapes by symbolic relationships.

Going from a graphical to an algebraic model of the times of sunset involves changing from one form of mathematical representation to another. Different representations imply different viewpoints—a different type of model represents a situation in a different way. A graphical model stresses relationships in geometric terms, telling a story using shape, size and position. An algebraic model stresses relationships in terms of symbols, telling a story using numbers, letters and mathematical functions.

Now work through Section 15.2 of Chapter 15 of the Calculator Book.

Activity 18 Keeping up with your learning

So far, this unit has involved you using ideas you have met earlier in the course and developing those ideas by integrating new work. Look at your Learning File to see which ideas/activities you have concentrated on.
How did you check your understanding of earlier work?
Did you need to look again at the calculator work you have been doing?
How did you get on using the mathematical terms?
Have you extended or revised earlier Handbook entries? (If you have not revised your earlier Handbook entries, look back at your earlier notes on trigonometric functions and add the ideas from this section to consolidate your work in *Units 9, 13* and *14*.)

Think about how you made the choices you did in working through this unit so far and record your thoughts on the Learning File sheet you used in Activity 1.

Outcomes

After studying this section, you should be able to:

- use the following terms accurately and be able to explain them to someone else: 'periodic behaviour', 'phase shift', 'phase lead', 'phase lag', 'identity' (Activities 10, 13 and 15);
- use your calculator to choose values for the amplitude, period or frequency and phase shift of a sine curve model, given a set of periodic data (Activity 8);
- outline the modelling steps you would take to fit a sine curve to a set of periodic data (Activity 12);
- relate the sine and cosine functions, using standard trigonometric identities (Activities 10, 13, 14, 15, 16 and 17);
- explain how to translate a sine or cosine curve vertically or horizontally, by adding an appropriate constant or phase shift (Activities 9 and 12);
- interpret and evaluate an expression of the form $M + A\sin(\omega t + \phi)$ (Activities 9, 10, 11, 14 and 15);
- use your calculator to explore parametric graphing.

3 Patterns of sound

Aims The main aim of this section is to show how periodic variations can be described by adding sine curves together. ◇

So far you have come across sine curves used to model smoothly varying periodic behaviour, and in their own right as mathematical objects. This section looks at sine curves as building blocks for sounds.

An important idea which crops up in technology and science as well as mathematics is that of adding sine curves of different amplitudes, frequencies and phases together to build up complex repeating patterns. Such patterns can look quite different from individual sine or cosine curves, but it turns out that almost any periodic shape can be built up just by adding together the right sinusoidal functions. A musical note produced by a musical instrument is a periodic wave made up of a combination of a fundamental frequency and higher frequency components.

However, the real power of this approach lies not in building periodic shapes up but in breaking them down. It is often useful to think of such shapes in terms of the collection of different sine curves that would have to be added together to describe them. This view offers a new way of talking and thinking about periodic behaviour.

Sine curves, sine waves and waveforms

In mathematics, the term 'sine curve' is used to mean the graph produced by a function such as $\sin x$, or $\sin 3x$. Similarly, the term 'cosine curve' refers to graphs produced by functions such as $\cos x$, or $\cos 4x$. In these expressions, x is simply the independent variable. Remember that sine and cosine curves are identical except for a phase shift of $\pi/2$ radians, or 90°. Anything expressed as a sine function can also be expressed as a cosine function, and vice versa.

In science and technology, you are likely to come across the term 'sine wave' to mean a sine variation with time. For example, the variation of air pressure with time produced by the vibrations of a tuning fork is modelled as a sine wave. A sine wave is represented by a function such as $\sin 2t$, or more generally as $A\sin(\omega t + \phi)$, where the variable t represents time. Sometimes, the general term 'sinusoid' is used to indicate either a sine or a cosine time variation.

The term 'waveform' is also common in science and technology. It is used to refer to the general shape of a time variation. The video associated with *Unit 9* showed waveforms associated with musical sounds displayed on the screen of an oscilloscope. In general, the term 'waveform' does not necessarily imply periodic behaviour.

3.1 The sound of silence

You should read the following notes and then watch the video band 'The sound of silence' and the short following piece 'Subtractin sound'. The total viewing time is some 19 minutes.

The video reviews some of the ideas about sine curves that you have met already, but it also looks forward and introduces some ideas that may be quite new to you. These ideas are discussed in more detail in this section, so do not worry if you do not understand everything the first time you watch the video. A good learning strategy is to watch all of the video now to review some basic ideas and get a flavour for what is to come. If you have time when you have completed Section 3, then you should watch the video again.

'The sound of silence' illustrates how a sine curve, sometimes called a *sine wave*, is related to circular motion. This section is intended to revise the ideas discussed in Section 1 and to offer you some animated images.

When two sine waves with frequencies within a few hertz of each other are added, the result is a periodic wave whose amplitude rises and falls at a frequency equal to the difference between the frequencies of the two sine waves. These regular amplitude variations are called 'beats'. You will be able to hear the effect of beats and see how they are represented graphically.

Recall the piano tuner in the audiotape associated with *Unit 9* made use of beats in tuning a piano.

The video also looks at the complex sounds produced by musical instruments. Different instruments playing the same note produce sounds at the same pitch but with a different quality, or timbre. The sound produced by a trumpet, for example, is different from that of a flute. Differences in timbre between instruments can be described by treating the sound as if it were made up of a many different sine waves at different frequencies.

The effect of a phase shift on a sine wave is to shift it to the right or to the left along the horizontal axis. In Subsection 2.2, you saw how a phase shift of 180 degrees or π radians effectively turns the curve upside down. Shifting all the sine wave components of a sound by 180 degrees has the effect of turning the entire periodic waveform upside down. When this shifted wave is added to the original the result is zero. In other words, two sounds can be added together to produce nothing—quite literally the sound of silence.

Sound-cancellation techniques based on similar principles can be used to reduce the level of noise from machinery in certain industrial environments.

The short video piece 'Subtractin sound' looks at the effect of removing the higher frequencies present in sounds. A sound is softened when higher frequency components are removed. The human voice can be thought of as being made up of a range of vocal sounds at different frequencies. Removing the higher frequency components changes the characteristics of a voice. You have almost certainly come across this effect on the telephone. Voices can sound 'restricted' and 'tinny' and in extreme cases you may not be able to recognize the other person's voice at all. The video explains how this effect occurs.

Now watch band 6 of Videotape 2 and complete the following activity.

Activity 19 Video notes

As you watch the video, make some notes and sketches so that you could explain the following ideas to someone who is not taking this course.

(a) What is the relationship between motion in a circle and the shape of a sine curve? Explain the terms 'amplitude', 'period', 'frequency' and 'phase shift'.

(b) What is a 'beat'?

3.2 Beats

Recall that equal temperament means that each octave is divided into 12 equal intervals, called *semitones*.

In the audiotape associated with *Unit 9*, you heard how a piano tuner used the acoustic phenomenon of 'beats' to tune a piano to equal temperament. In musical terms, a beat is a pulsing rhythm which can occur when two strings are vibrating at different frequencies. The beats disappear when the strings are tuned to the same frequency, or when their frequencies are in some special relationship to each other, such as an octave, a perfect fourth or a perfect fifth. This subsection looks at how beats can be represented mathematically.

In *Unit 9*, you also saw how a pure musical tone can be represented by a sine function. When two such tones are sounded together, their combined effect can be modelled by adding together the corresponding sine functions.

Frequencies of 6 and 7 Hz are rather too low to hear, but are easier to show in a diagram. Beats between tones of higher frequencies follow the same principles.

Figure 21(a), opposite, shows two sine waves representing pure tones at the frequencies 6 Hz and 7 Hz. The amplitudes of the tones are equal. Over a time interval of one second, the 6 Hz sine wave goes through 6 whole cycles and the 7 Hz sine wave goes through 7 whole cycles. Suppose that at one particular instant both waves pass through zero on their way towards their peak positive value; at this point they are said to be in *phase* with each other.

▶ What happens next?

As time increases the waves move out of phase with each other. After 0.5 seconds the 6 Hz tone has completed 3 cycles while the 7 Hz tone has completed 3.5 cycles. Both waves are again passing through zero together, but this time one is going from positive to negative on its way to a negative peak, while the other is going from negative to positive on its way to a positive peak. They are now completely out of phase. Over the next 0.5 seconds the curves move back into phase again so that at the end of the interval both waves are passing through zero again, going in the same direction towards a positive peak.

Figure 21(b) shows the result of adding the two sine waves together.

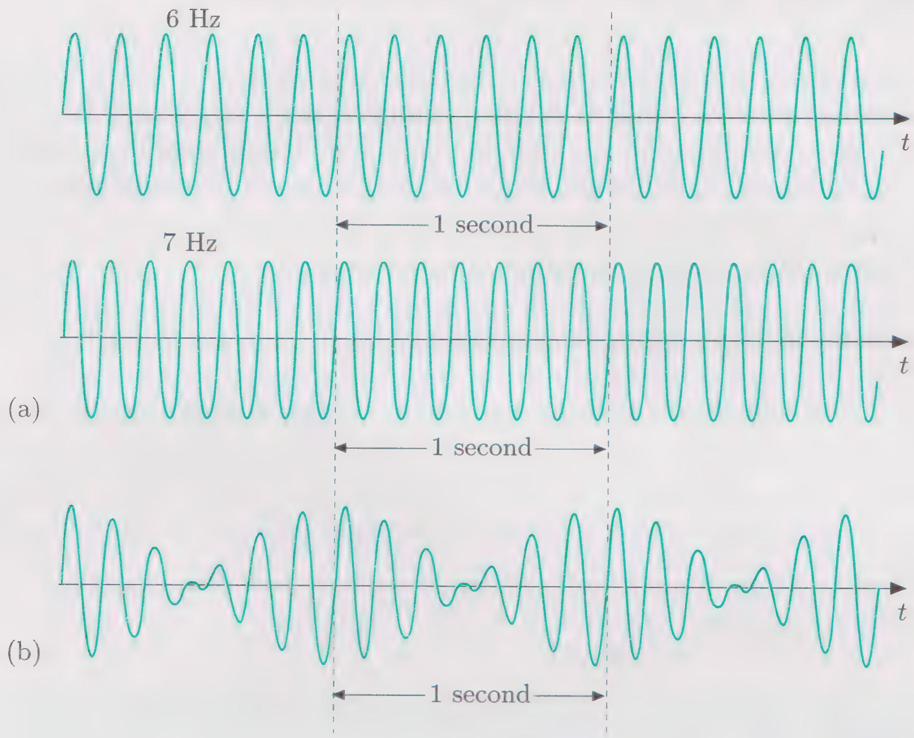


Figure 21 (a) 6 Hz and 7 Hz sine waves and (b) their sum

No vertical scale is shown: focus on the shape here.

Figure 22 shows how part of the shape is formed. When both waves are positive, their sum is also positive; similarly when both are negative, their sum is negative. When one is positive and the other negative however, the waves can add to give either a positive or a negative sum. When the two waves have equal values, but opposite signs, they add to give zero.

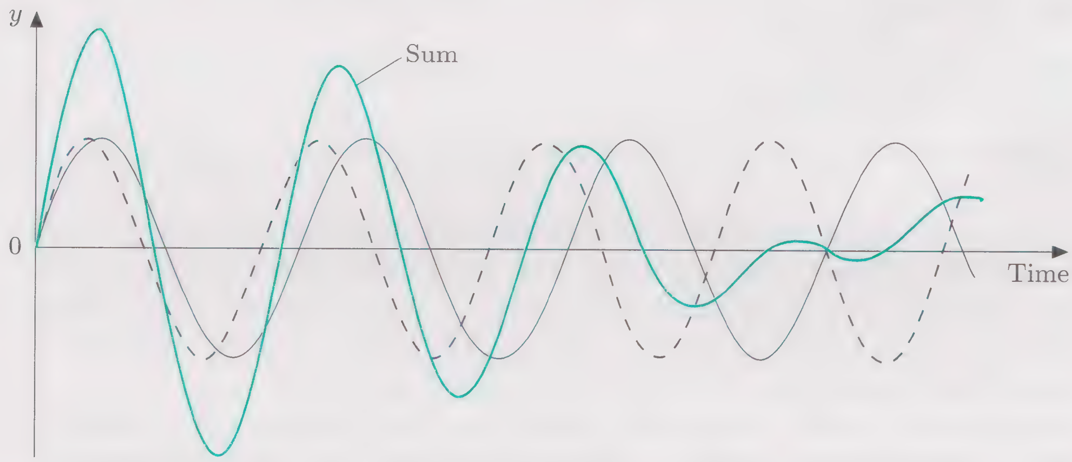


Figure 22 Adding two sine waves together

Notice the sum is twice the height of the individual waves when completely in phase.

The sum of the two sine waves appears to be another periodic graph but one whose amplitude varies periodically. The amplitude peaks when the 6 Hz and 7 Hz waves are in phase, but reduces to zero when the waves are out of phase. As the waves move in and out of phase every second, this pattern repeats over and over again.

▶ What would an acoustic pattern like this sound like?

If the two tones are not of equal amplitudes, the beat phenomenon will not be so pronounced.

The amplitude of a sound wave determines the loudness of the sound. For beats between audible tones a listener would hear a tone whose loudness regularly rises and falls, giving the distinctive rhythmic 'wah-wah-wah' sound called 'beats'. In Figure 21, the beat pattern is repeated once every second. In general, the number of beats that occur each second is equal to the difference in frequency between the two tones.

▶ How are beats represented mathematically?

Frequencies of 6 Hz and 7 Hz correspond to angular frequencies of $6 \times 2\pi = 12\pi$ radians per second, and $7 \times 2\pi = 14\pi$ radians per second respectively. The 6 Hz tone can be represented by $\sin 12\pi t$, and the 7 Hz tone by $\sin 14\pi t$ and the effect of sounding them together is modelled by the sum:

$$\sin 12\pi t + \sin 14\pi t$$

Activity 20 The beat goes on

Enter the functions $\sin 12\pi t$ and $\sin 14\pi t$ on your calculator and display both curves together over the interval $t = 0$ to $t = 1$ (second). Notice how they start from the same point, move out of phase and then back into phase to finish together after 1 second.

Display the curve of the function $\sin 12\pi t + \sin 14\pi t$ on its own. You will get a better view of the beat pattern if the curve is plotted over the interval $t = 0$ to $t = 2$ (seconds). Confirm that the amplitude reaches a peak once every second.

What would be the period of the beat if the frequencies were 6 Hz and 8 Hz? Use your calculator to investigate.

Recall the discussion in Chapter 9 of the *Calculator Book*, on adding sine curves.

Adding sine curves can lead to graphs which are not sine curves, but which are periodic. However, the problem with the formula $\sin 12\pi t + \sin 14\pi t$ is that, although it is mathematically correct, it is not very helpful in telling you what sort of graph to expect. Unless you happen to know what the overall curve looks like—or have a calculator handy to do some experiments—it tells you nothing about how often beats occur, or what sort of mathematical function governs the overall shape, or what happens *in general* when two sine waves are added. At this stage in the course, you know that going from the particular to the general is one of the characteristic features of mathematics.

In Section 2, you met the idea of trigonometric identities. Such identities are equivalent ways of writing down sine and cosine expressions. This idea is used again to look at an alternative way of expressing the sum of two sine curves.

The first step is to look at the example shown in Figure 21 again. The overall waveform looks like a sine wave, only one with a periodically varying amplitude. How might this idea be expressed mathematically?

You know that a general way of writing a sine function with an amplitude A and a frequency f is just $A \sin(2\pi f t)$. This is just the product of two parts: a number A and a sine curve with an amplitude of 1. All that has been done is to write $A \sin(2\pi f t)$ as $A \times \sin(2\pi f t)$.

Now look at Figure 23. You can see the graphs for A and $\sin(2\pi f t)$, and the product $A \times \sin(2\pi f t)$, both plotted against t, representing time.

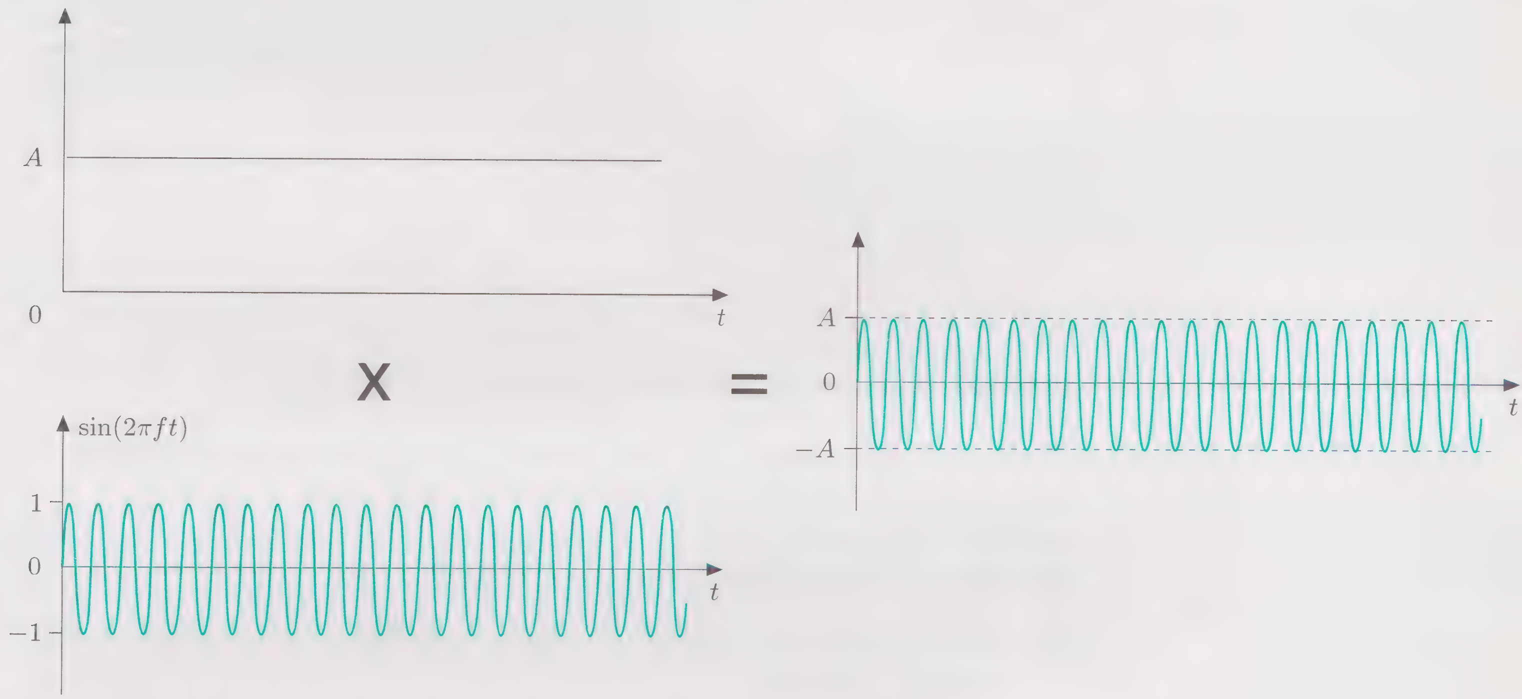


Figure 23 Graphically forming the product $A \times \sin(2\pi f t)$

Each point on the sine curve has been multiplied by the number A, scaling the curve so that the product reaches peaks of A and $-A$, rather than 1 and -1. Notice that A is a constant: it has the same value for all values of t.

But A does not have to be a constant; the graph of A does not have to be a horizontal straight line. The value of A could depend on t as in Figure 24, overleaf. Here the graph of A against t is a straight line with an intercept of 1.

Now if the values on the graphs of A and $\sin(2\pi f t)$ are multiplied together, the resulting waveform looks like a sine wave whose amplitude is steadily growing with time. As the value of A increases with t, so does the peak amplitude of the sine wave. Because the value of A multiplies both positive and negative values of the sine function, the positive and negative parts of the curve grow together.

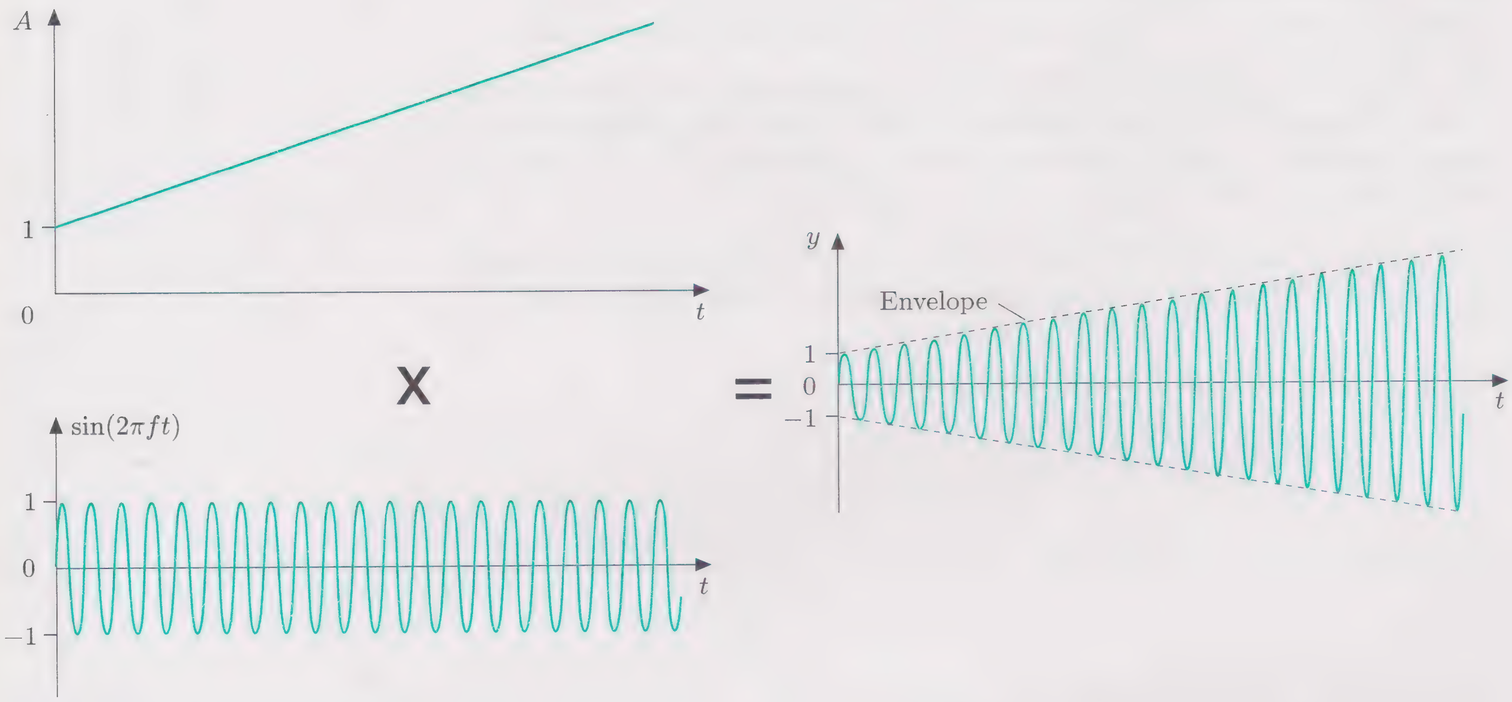


Figure 24 Graphically forming the product $(1+t) \times \sin(2\pi f t)$

The dashed line joining the peaks of the growing sine curve is called the *envelope* of the curve; you can think of the curve as being entirely contained within the envelope. The shape of the envelope is determined by the shape of the graph of A.

▶ What is the mathematical expression that describes the growing sine curve in Figure 24?

If necessary, look back to *Units 10* and *13* to remind yourself about straight-line relationships.

The graph of A against t is a straight line with an intercept of 1 and a slope of 1, so the formula for the straight line is:

$$A = t + 1 = 1 + t$$

The formula for the sine curve is:

$$y = A\sin(2\pi f t)$$

So replacing A by the formula for the straight line gives the expression for the growing sine curve:

$$y = (1+t)\sin(2\pi f t)$$

Activity 21 Growing sines

What is the general formula of a straight line? Use y and t as variables.

Write down the formula for the growing sine curve whose envelope is defined by a straight line of intercept 1 and slope 0.5. Use your calculator

to display the curve if the frequency f associated with the sine function is 1 Hz.

What would the curve look like if the slope of the envelope were -0.5? Use your calculator to investigate.

The formula for A does not have to describe a straight line. Figure 25 shows the case where the graph of A varies sinusoidally. Multiplying the function $\sin(2\pi ft)$ by this function gives, using the same reasoning as before, a periodic curve whose envelope varies sinusoidally. Notice again that the shape of A specifies the boundary of both the positive and the negative parts of the curve.

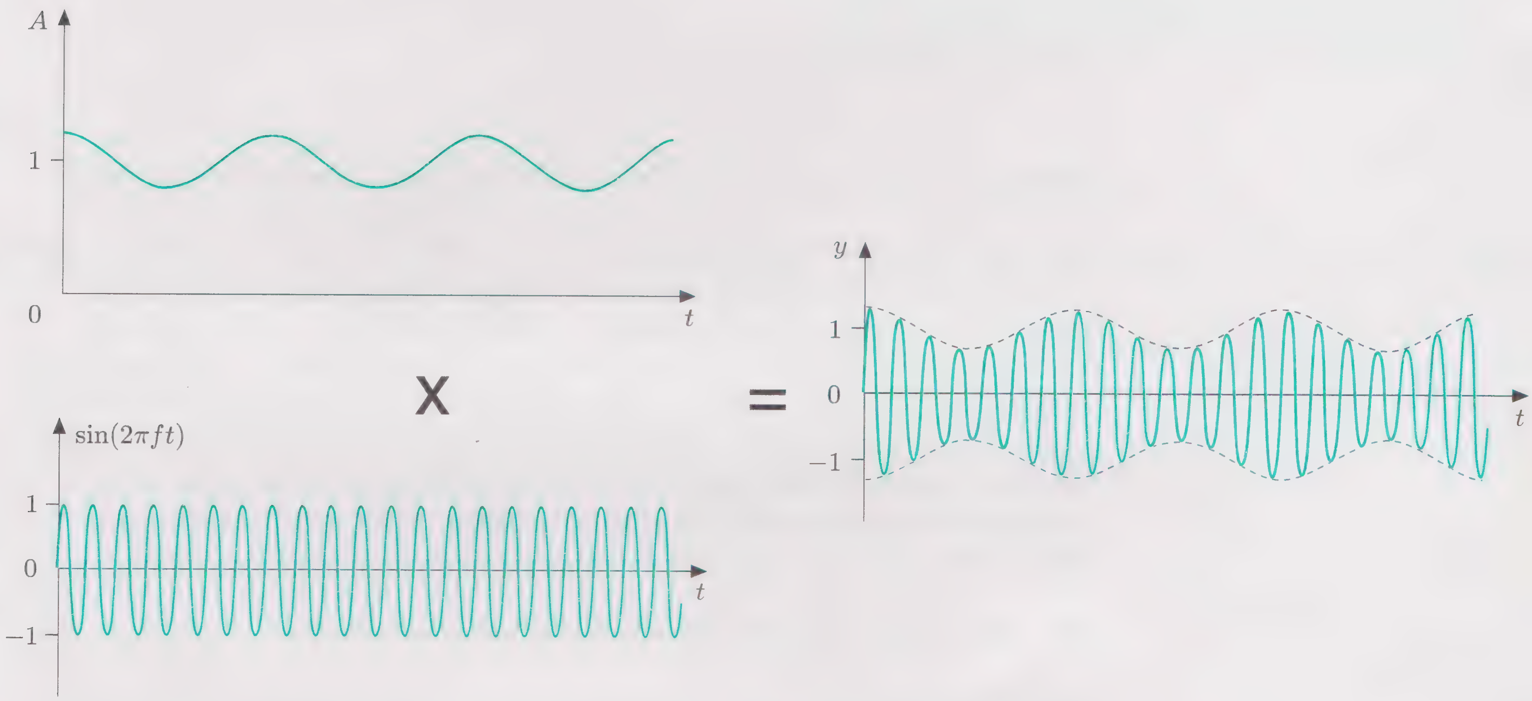


Figure 25 A sinusoidal variation in amplitude

In the last activity, you investigated what happened when the straight-line graph of A had a negative gradient. You should have found that the value of A, and hence the value of the product $(-0.5t + 1)\sin(2\pi t)$ became smaller as t increased, reaching zero at $t = 2$ seconds. A then became negative. As A becomes increasingly negative, the envelope of the sine curve started to grow again.

You can apply the same sort of reasoning to Figure 26.

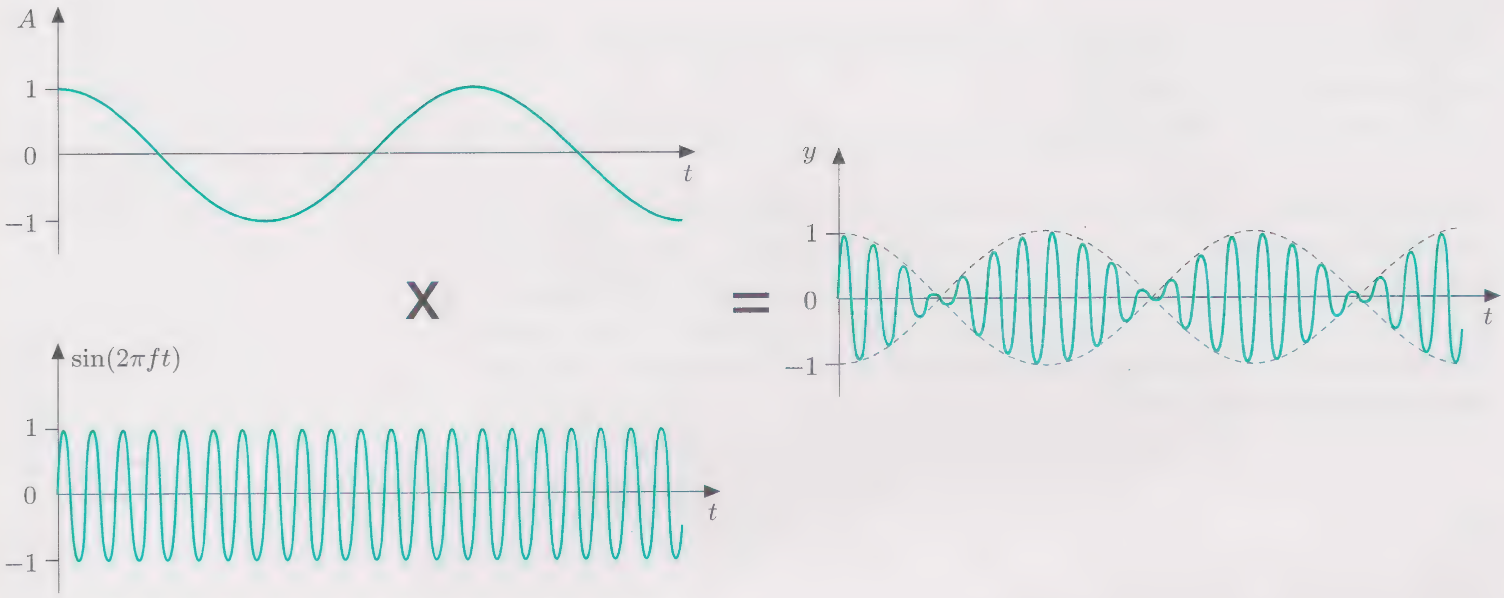


Figure 26 Graphically forming the product of a cosine curve and a sine curve

Recall that a cosine curve has exactly the same shape as a sine curve except that it starts at 1 rather than zero.

Here the graph of A plotted against t is a cosine curve. In comparison with the sine curve it is varying relatively slowly. You can see the result of multiplying the two curves together. Now the envelope of the resulting periodic curve is defined by the slowly varying cosine curve. Look carefully at how the envelope varies. It starts at a peak when the value of the cosine is 1 (at $t = 0$), reduces to zero as the cosine goes through zero, and grows to a peak again as the value of the cosine reaches -1. This cycle then repeats as the cosine goes through zero again on its way back to 1. In other words, the envelope has two peaks for each cycle of the cosine curve.

▶ How would this curve be described mathematically?

The formula for A is now a cosine function. If the frequency of the cosine function is represented by g, then the formula for A is $\cos(2\pi gt)$. So the formula for the periodic curve is just the product of the cosine and the sine functions:

$$y = \cos(2\pi gt) \times \sin(2\pi ft)$$

The curve described by this formula is the same general shape as that of the sum of two sine curves, producing beats. Figure 27 shows again the beats produced by the combination of a 6 Hz and a 7 Hz pure tone as shown in Figure 22. The envelope, which will be produced in the model by the $\cos(2\pi gt)$ term in the formula above, is shown as a dashed line.

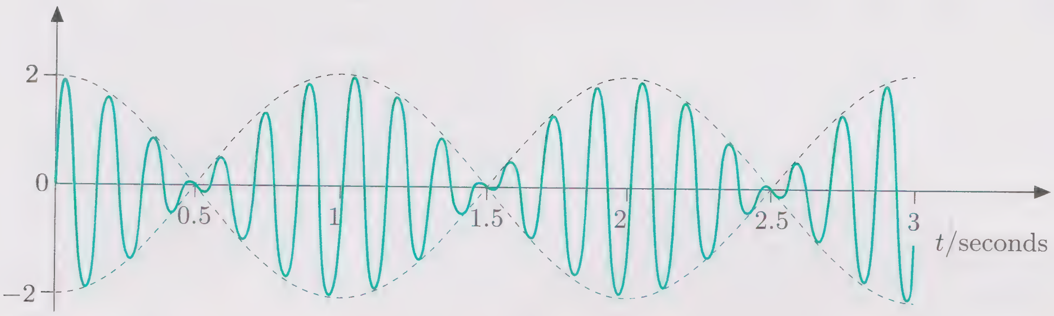


Figure 27 Redrawing of Figure 22 data: the dashed line shows the envelope of the 1 Hz beat

▶ What is the frequency g corresponding to this envelope?

To answer this, you need a link between the acoustic phenomenon of beats and the mathematical model.

Acoustically, the combination of the 6 and 7 Hz tones produces one beat every second. In the model, one cycle of the cosine function produces two amplitude peaks, corresponding to two beats. So one cycle of the cosine function corresponds to an interval of 2 seconds. In other words, the period of the cosine function is twice the period of the beats. In terms of frequency, the frequency of the cosine wave must be half the frequency of the beats. In this case, the beat frequency is 1 Hz (1 beat per second), so the frequency g of the cosine function is 0.5 Hz. Hence the shape of the envelope of the beat waveform is described by the term:

$$\cos(2\pi \times 0.5 \times t) = \cos \pi t$$

So far the peak amplitude of the cosine envelope has been taken to be 1. But if you look again at Figure 27, you will see that the peak value of the envelope is 2. So the envelope of the new model must be scaled by 2 so that it matches the original. Multiplying the cosine envelope by 2 gives the general model:

$$y = 2\cos(2\pi g t)\sin(2\pi f t)$$

In the current example, the two tones differ in frequency by 1 Hz, so $g = 0.5$ Hz and the model is:

$$y = 2\cos(\pi t)\sin(2\pi f t)$$

But you can go a little further with this model by bringing to bear what you know about beats. You know that for tones of 6 Hz and 7 Hz, beats occur at a frequency that is the difference between the frequencies: that is, at $7 - 6 = 1$ Hz. In general, if f_1 represents the frequency of one tone and f_2 represents the frequency of the other, then beats occur at the difference frequency $f_1 - f_2$. But you also know that the frequency of the cosine envelope in the model is half the beat frequency, that is $g = (f_1 - f_2)/2$. So you can write the model in a more general form:

$$y = 2\cos\left[2\pi\left(\frac{f_1 - f_2}{2}\right)t\right]\sin(2\pi f t)$$

This model now describes the envelope—that is, the overall shape of the beat waveform for any two frequencies where beats occur. It looks rather more complicated than before but the essence is straightforward. For two tones of frequencies f_1 and f_2, the beat pattern is a sine curve of some frequency f (about which more will be said soon) whose amplitude varies sinusoidally from 0 to 2 (described by the term $2\cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t$). Beats are produced at a frequency equal to the difference between the frequencies of the tones.

The term 'sinusoidal' is used to talk about both sine and cosine variations.

Activity 22 Bounding the beat

Use your calculator to plot the function $\sin 2\pi f_1 t + \sin 2\pi f_2 t$ for the frequencies $f_1 = 7\,\text{Hz}$ and $f_2 = 6\,\text{Hz}$, over a time scale of 0 to 2 seconds.

While displaying the beat pattern, also plot the envelope function $2\cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t$ for these frequencies and see how the cosine curve determines the boundary of the beat pattern.

Repeat the exercise for the frequencies 12 Hz and 16 Hz. What is the beat frequency in this case, and what is the cosine envelope function?

Recap for a moment. You started with an observation that when two tones of different frequencies f_1 and f_2 are sounded together, they can produce beats. Beats occur at the difference frequency $f_1 - f_2$. A mathematical model describing the combination of the tones is:

This is sometimes called a *linear combination,* because it is assumed that the effect of the two tones sounding together is modelled by adding the corresponding sine curves together.

$$y = \sin 2\pi f_1 t + \sin 2\pi f_2 t$$

The shape of the graph produced by this model is that of a sine curve whose amplitude varies sinusoidally at a frequency equal to half the beat frequency. A more appropriate model, therefore, is one which is more directly related to this description, such as the expression:

$$y = 2\cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t\ \sin 2\pi f t$$

What this model does is to relate the shape of the envelope of the overall curve to the periodic pattern of beats. But nothing has been said so far about the frequency f of the underlying curve $\sin 2\pi f t$.

The two mathematical models are different ways of expressing the same physical behaviour. The aim is to build a new identity which relates the sum of two sinusoidal curves to the product of two sine curves. So far this identity has the form:

$$\sin 2\pi f_1 t + \sin 2\pi f_2 t = 2\cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t\ \sin 2\pi f t$$

The next step is to relate the frequency f to the frequencies f_1 and f_2 of the two original tones.

You have concentrated on the envelope of the curve. Now concentrate on what is happening inside the envelope, on the details of the underlying sine curve. Before you go further, pause a moment and think about what you might expect to occur inside the envelope. What is going on physically? Two tones are sounding together, and what you hear is another tone whose loudness is varying. You interpret the variation of loudness as beats, but what is the frequency of the tone? Is its pitch likely to be higher than either of the two original tones—or lower—or somewhere in between? How could you find out?

Expectations and good guesses all help to inform mathematical insight. You often have to try things out to see which way to go.

If you have a good musical ear you might be able to tell the pitch just by listening to the overall sound, but the effect of the beats can make this difficult. Another way is to measure the frequency, using suitable equipment, but again this is not the sort of thing everyone has access to. What is going to be done is something of a compromise: the original mathematical model will be used in lieu of a real sound (the aim, after all, is to find an identity between different models).

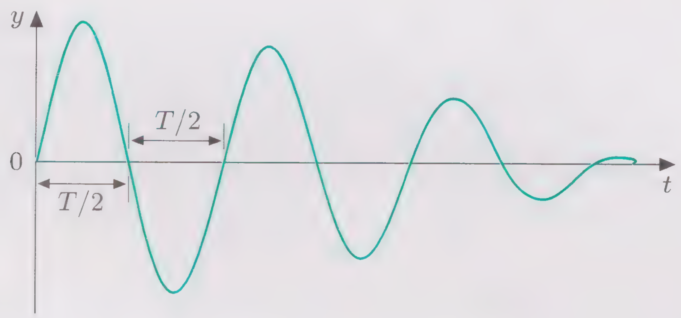


Figure 28 Part of the beat waveform

Figure 28 shows the first 0.5 seconds of the curve for the model:

$$y = \sin 12\pi t + \sin 14\pi t$$

representing the beat between the 6 Hz and 7 Hz tones. Taking this as a sine curve of decreasing amplitude, you can measure the time equivalent to the 'period' of the sine curve and hence work out its equivalent 'frequency'. You can do this on your calculator: first plot the curve and then measure the time between the 'zero-crossings'—the points where the curve crosses the horizontal axis where $y = 0$. The time between two consecutive zero-crossings is equivalent to half the period. Check that the zero-crossings occur at equal intervals of time (as far as it is possible to tell on your calculator).

The curve inside the envelope is not strictly a sine curve, because its amplitude varies with time, so terms such as 'period' and 'frequency' have to be used carefully.

Activity 23 *A hidden pattern?*

Use the graphing and trace facilities on your calculator to find the 'period' and 'frequency' associated with the curve inside the envelope of the following function.

$$y = \sin 12\pi t + \sin 14\pi t$$

Repeat this activity for sine curves representing tones of 9 Hz and 11 Hz, and 14 Hz and 18 Hz. Can you find a link between the frequencies of the individual tones and the 'frequency' of the curve inside the envelope?

Looking for patterns by trying out a few particular examples can suggest how things might go in general.

Working out individual examples is not any sort of mathematical proof, but the previous activity should have suggested to you that there might be a pattern to the relationship between the frequencies. In fact, it turns out that the curve within the beat envelope corresponds to a tone with a frequency that is the average, or arithmetic mean, of the two original tones. That is, in the model:

$$y = 2\cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t \ \sin 2\pi f t$$

the frequency f in the sine term is equal to $(f_1 + f_2)/2$. The final mathematical model is therefore:

$$y = 2\cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t \ \sin 2\pi\left(\frac{f_1 + f_2}{2}\right)t$$

This rather formidable expression is the mathematical description of the beat waveform produced by two tones sounding together. It is the product of a cosine curve with a frequency equal to half the difference between the tones (producing beats at the difference frequency), and a sine curve with a frequency equal to the arithmetic mean or average value of the tones. Multiplying by the cosine term affects the amplitude of the underlying sine term.

So now you have two mathematical expressions modelling the same thing—the original sum of two sines and the new product formula. The two expressions give a new mathematical identity:

$$\sin 2\pi f_1 t + \sin 2\pi f_2 t = 2\cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t \ \sin 2\pi\left(\frac{f_1 + f_2}{2}\right)t \qquad (1)$$

From a mathematical point of view, this identity is so far only a suggestion. All you have done is a few numerical investigations at some particular frequencies. You have not proved in a way mathematicians would accept that this expression holds true for all possible values of f_1 and f_2, and it is beyond the scope of this course to do so. But it turns out that the identity *is* true; in fact, it is part of a large group of formulas involving relationships between sines and cosines. If you take your mathematics studies further, you are likely to come across other examples.

Activity 24 Identity crisis

How would you boost your confidence that identity (1) is true—that the left-hand side is always equal to the right-hand side for *any* values of f_1, f_2 and t? You cannot try every value, but a useful approach is to try out some special—and easy—cases to see if the two sides come out the same. What do you get in the following cases?

(a) $t = 1$ second, $f_1 = 1\,\text{Hz}$, $f_2 = 1\,\text{Hz}$.

(b) $t = 0$, $f_1 = 0\,\text{Hz}$, $f_2 = 1\,\text{Hz}$.

(c) $t = t$ and $f_1 = f_2 = f$.

The argument about beats was developed using frequencies expressed in hertz. But you could express identity (1) using angular frequency, or a more general form, instead.

Activity 25 Forms of identity

(a) Recall that angular frequency ω (in radians per second) is related to frequency f (in hertz) by the formula $\omega = 2\pi f$. Rewrite identity (1) using angular frequency.

(b) In mathematics books, you are likely to see the identity written in its most general form. Write down this form by replacing $2\pi f_1 t$ (or $\omega_1 t$) by the symbol a, and $2\pi f_2 t$ (or $\omega_2 t$) by the symbol b.

3.3 Fourier's idea

The sounds made by musical instruments are much more complex than beats but, for a sustained note, they are still periodic. Figure 29 shows two examples. In (a), you can see the waveform produced by a trumpet sounding the note A above middle C, and in (b), the waveform produced by a flute sounding the same note. Recall from *Unit 9* that the frequency of A above middle C is 440 Hz (concert pitch). Although the detailed shapes of the two waveforms are different, both are periodic, and both have the same period of $1/440 = 0.00272$ seconds, or about 2.3 milliseconds.

Recall that in the SI system of units a millisecond is 0.001 seconds, or one thousandth of a second.

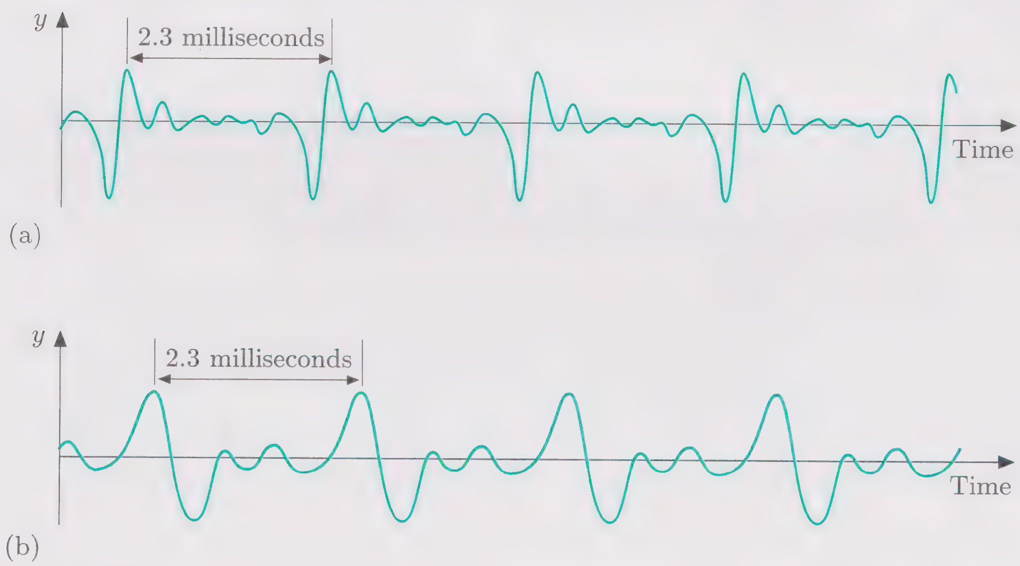


Figure 29 Waveforms produced by (a) a trumpet and (b) a flute

▶ How are periodic waveforms like these to be described?

In the early part of the nineteenth century, Jean-Baptiste Joseph Fourier developed a method, now known as Fourier analysis, which has enormous significance in many areas of mathematics, science and technology.

Historical note

Jean-Baptiste Joseph, Baron de Fourier (1768–1830) was involved in French revolutionary politics, accompanied Napoleon on his Egyptian campaign in 1798 and become a leading civil administrator and secretary to the Egyptian Institute, which Napoleon founded. In 1801, he returned to France with the responsibility of publishing the enormous volume of research on Egyptian antiquities carried out by the Institute. He became Préfet (senior administrator) of the Department of Isère at Grenoble and during this time also worked on problems in mathematical physics. In 1822, he published his work on the theory of heat, where he used combinations of sine and cosine functions to model heat flow.

The basis of Fourier analysis is that any periodic curve can be thought of as if it were made up of sine curves with frequencies related in a particular way. Start by looking at a periodic—but somewhat unmusical—waveform. This general shape, shown in Figure 30, is usually called a square wave. Waveforms like this turn up in areas such as communications, digital systems, and the electronic production of sounds.

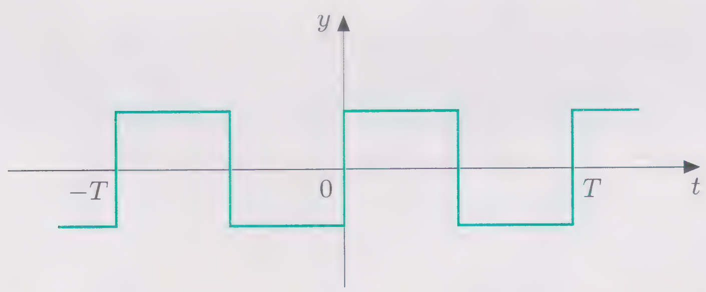


Figure 30 Square wave

▶ How would you describe this shape mathematically?

The period of the square wave is represented by T, corresponding to a frequency $f = 1/T$. Fourier analysis tells us that you can build up this waveform by adding together a number of different sine waves. Look at Figure 31 opposite. Here you can see how the shape of the square wave is built up.

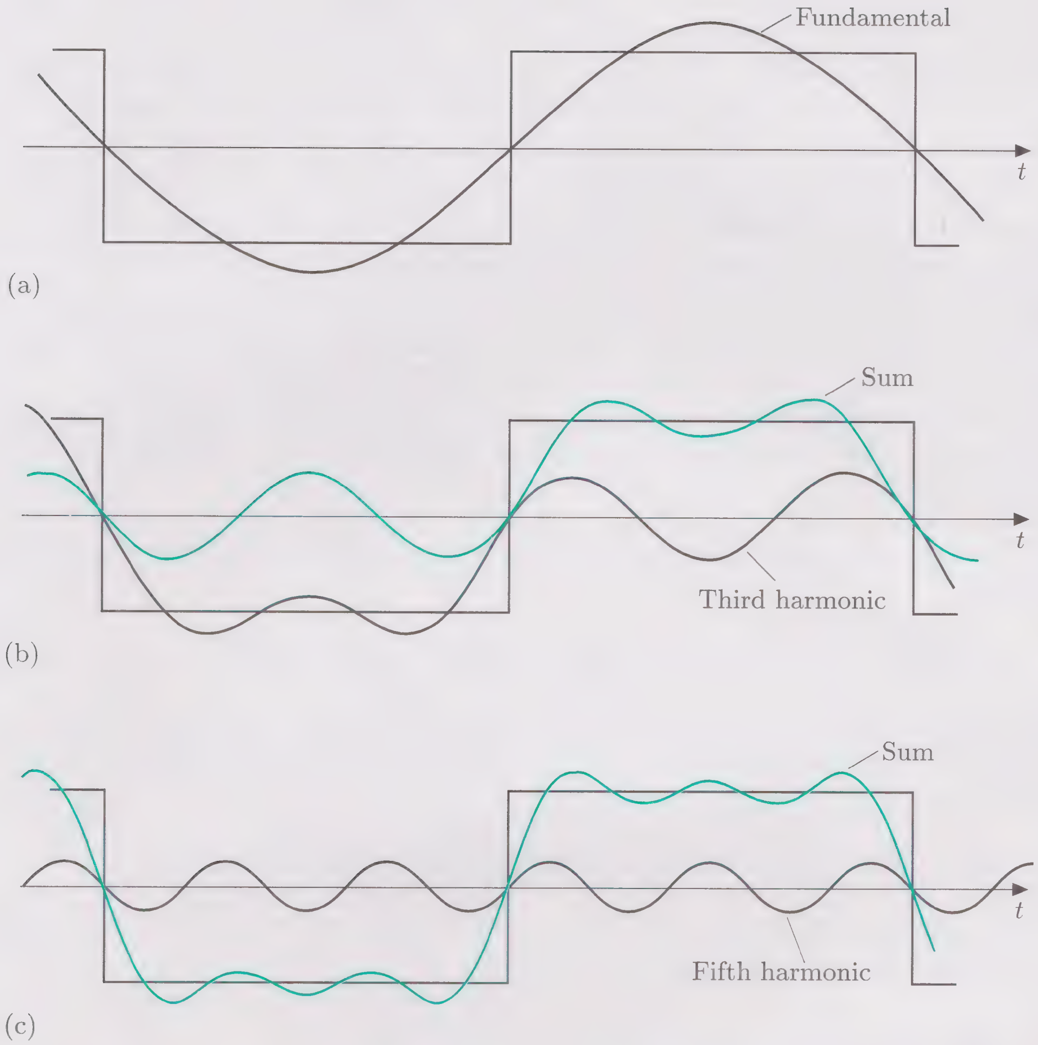


Figure 31 Approximating a square wave by (a) the fundamental, (b) adding the third harmonic, (c) adding the fifth harmonic

In Figure 31(a), a single sine wave with the same period as the square wave offers the first approximation to it. The frequency $f = 1/T$ of this sine wave is called the *fundamental* frequency; the sine wave itself is called the *fundamental component.*

Figure 31(b) shows the sum of the fundamental component and a sine wave with a frequency of three times the fundamental frequency. This new component is called the third harmonic. Notice that the amplitude of the new component is less than the fundamental. The sum of the fundamental and the third harmonic produces a waveform which has the same period as the square wave, but which is a better approximation to it than the fundamental alone.

Figure 31(c) shows the sum obtained by adding a further sine wave. This time its frequency is five times that of the fundamental, and its amplitude is smaller still. Adding in this fifth harmonic improves the approximation to the square wave even more.

If you go on to higher level courses in mathematics, technology or science you are likely to meet Fourier analysis again.

This process does not have to stop with the fifth harmonic. If you continue to add harmonic sine wave components of increasing frequency and diminishing amplitude, you continue to improve the approximation to the square wave. The mathematical details of how the amplitudes and frequencies of the sine wave harmonic components needed to make up a particular periodic shape are worked out is beyond this course. But the general interpretation of the results is relatively straightforward. The square wave can be thought of as being made up of the sum of a fundamental sine wave at a frequency f (where $f = 1/T$, and T is the period of the square wave) and harmonic sine waves at the frequencies $3f$, $5f$, $7f$, $9f$, and so on. It also turns out that the amplitude of each harmonic component of the square wave is inversely proportional to its frequency. Thus, the amplitude of the component at $3f$ is $1/3$, at $5f$ it is $1/5$, and so on.

The end result is that the square wave can be expressed by the sum of sine waves:

$$\sin 2\pi ft + \frac{1}{3}\sin 2\pi(3f)t + \frac{1}{5}\sin 2\pi(5f)t + \frac{1}{7}\sin 2\pi(7f)t + \cdots$$

Sometimes the fundamental frequency f is written as f_1 to show that it is a specific frequency, and not a general variable which could take on any value:

$$\sin 2\pi f_1 t + \frac{1}{3}\sin 2\pi(3f_1)t + \frac{1}{5}\sin 2\pi(5f_1)t + \frac{1}{7}\sin 2\pi(7f_1)t + \cdots$$

Using angular frequency, you can replace $2\pi f_1$ by ω_1, $2\pi(3f_1)$ by $3\omega_1$, and so on, and write the sum of sine waves as:

$$\sin \omega_1 t + \frac{1}{3}\sin 3\omega_1 t + \frac{1}{5}\sin 5\omega_1 t + \frac{1}{7}\sin 7\omega_1 t + \cdots$$

This sum is called a *Fourier series.* Each term in the series makes a contribution to the approximation of the square wave.

Activity 26 Components of a square wave

A particular square wave has a period of 0.5 second.

(a) What are the frequencies in Hz of the fundamental and harmonic components of the wave?

(b) Write down the Fourier series of the square wave.

(c) Use your calculator to display the fundamental component, then add in the next component and display the result, and so on for the first four components. What shape do you see building up?

A different mix of harmonic components can change the waveshape significantly. A waveform known as a sawtooth wave, for example, uses the fundamental ω_1 and *all* the harmonics (not just the odd numbered ones) and has the Fourier series:

$$\sin \omega_1 t + \frac{1}{2} \sin 2\omega_1 t + \frac{1}{3} \sin 3\omega_1 t + \frac{1}{4} \sin 4\omega_1 t + \frac{1}{5} \sin 5\omega_1 t + \cdots$$

As with the square wave, each new term improves the approximation to the sawtooth shape.

Activity 27 Components of a sawtooth wave

What is the period of the sawtooth wave if the value of the fundamental frequency ω_1 is 20 radians per second? Write down the Fourier series of the waveform.

Use your calculator to display the waveform you get with the first four terms of the Fourier series.

In Subsection 2.2, you saw that shifting the phase of a sine wave by π radians (180°) effectively turned it upside down. The identity is:

$$\sin(x + \pi) = -\sin x$$

Introducing a phase shift of π gives the same result as multiplying the sine function by -1. If you add a sine wave and its phase shifted version the result, not surprisingly, is always zero:

$$\begin{aligned} \sin x + \sin(x + \pi) &= \sin x - \sin x \\ &= 0 \end{aligned}$$

This result can be extended to the Fourier series of a periodic waveform. If you shift the phase of each sine wave component in a Fourier series by π radians, then the result is the same as turning the entire waveform upside down or, what is equivalent, multiplying it by -1. For example, shifting each component in the Fourier series of a sawtooth waveform by π radians gives:

$$\sin(\omega_1 t + \pi) + \frac{1}{2} \sin(2\omega_1 t + \pi) + \frac{1}{3} \sin(3\omega_1 t + \pi) + \frac{1}{4} \sin(4\omega_1 t + \pi) + \cdots$$

which is the same as:

$$-\sin \omega_1 t - \frac{1}{2} \sin 2\omega_1 t - \frac{1}{3} \sin 3\omega_1 t - \frac{1}{4} \sin 4\omega_1 t - \cdots$$

or

$$-\left(\sin \omega_1 t + \frac{1}{2} \sin 2\omega_1 t + \frac{1}{3} \sin 3\omega_1 t + \frac{1}{4} \sin 4\omega_1 t + \cdots\right)$$

which is the original series multiplied by -1. If you add this series to the original series the sine terms will cancel each other out and you will be left with nothing. If the Fourier series represents a sound wave, the result of adding two sounds whose frequency components are shifted in phase by π radians relative to each other will be silence.

This effect was demonstrated in the video. The technique is used in some noise-cancelling applications.

Activity 28 The sound of silence

On your calculator display the sawtooth wave from Activity 27.

Now enter the Fourier series of the sawtooth wave again, this time adding a phase shift of π radians to each sine wave component and display the original and the new waveforms together. What do you see?

Finally display the sum of the two waveforms. What do you see now?

Now work through Sections 15.3 and 15.4 of Chapter 15 of the Calculator Book.

3.4 The spectrum

The Fourier series offers a new way of thinking about periodic waveforms. Any periodic waveform can be expressed as a sum of harmonic frequency components. This sum is unique: any particular waveform is associated with one and only one range of harmonic components. This leads to the idea that a particular set of frequency components is characteristic of a particular waveform.

The range of frequency components that characterizes a particular waveform is called the *frequency spectrum* of the wave. You may well have come across the idea of a spectrum in other contexts. For example, the 'visible spectrum' means the range of colours from red to violet to which our eyes respond. Sunlight can be thought of as being made up of this spectrum of colours—in fact, in a rainbow you see the effect of droplets of water splitting the light up into a band of colours. Light is an electromagnetic wave with some properties very much like the waves just discussed. Each colour corresponds to light of a particular frequency. In a rainbow, you see the visible spectrum of sunlight from the lower frequency red light at one end to the higher frequency violet light at the other.

You will learn more about the mathematics of the rainbow in *Unit 16*.

At its most basic, a spectrum is a range of frequencies. A graphical way of representing a spectrum is shown in Figure 32. The horizontal axis represents frequency and the vertical axis represents amplitude. On this plot, a single sine wave with an amplitude A and a frequency f is indicated as a vertical line of height A at a position f on the frequency axis.

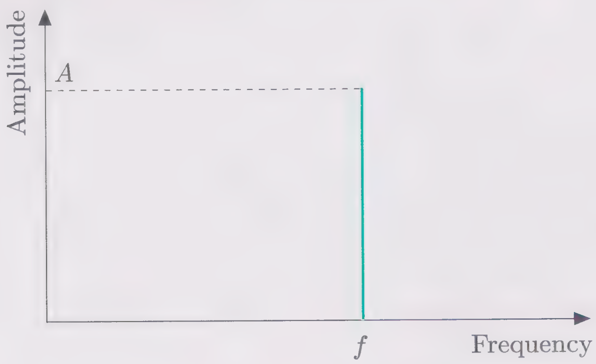


Figure 32 Spectrum of a single sine wave

More complex waves have a more complex spectrum. In Subsection 3.2, you saw how beats are formed between two tones whose frequencies differ by just a few hertz. Each tone is modelled as a sine wave and each sine wave can be thought of as a frequency component of the beat waveform. Figure 33 shows the frequency spectrum of the beat formed by a 6 Hz tone and a 7 Hz tone. There are two lines, one corresponding to the 6 Hz sine wave and the other corresponding to the 7 Hz sine wave. The heights of the lines are equal showing that the amplitudes of the sine wave components are equal.

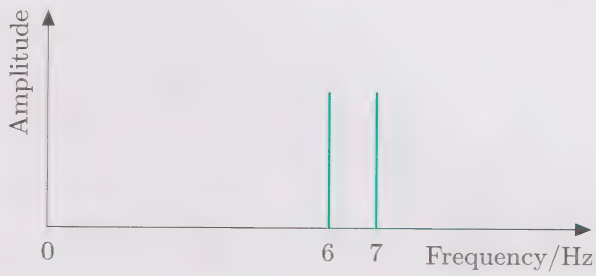


Figure 33 Spectrum of a beat pattern

Figure 34 shows the spectrum for the square wave. The line corresponding to the fundamental frequency is shown with a height of 1 at a frequency f. The harmonic components at frequencies $3f$, $5f$, $7f$, $9f$, and so on, are represented by lines with heights 1/3, 1/5, 1/7, 1/9 respectively. The amplitudes of the high-frequency components of the square wave get smaller and smaller as the frequency increases, but they never vanish completely.

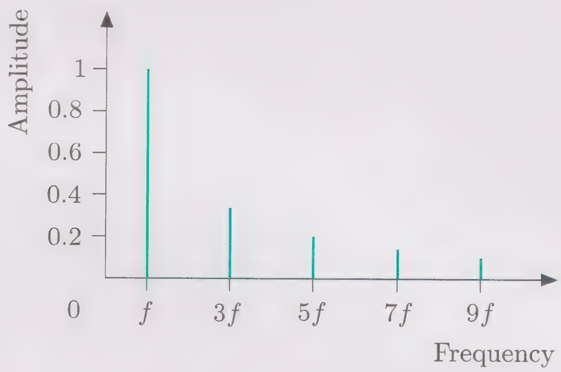


Figure 34 Spectrum of a square wave

The important point about the frequency spectrum of a periodic waveform is that each line represents a sine wave. The position of the line along the horizontal axis indicates the frequency of the sine wave. On a plot of a frequency spectrum, the lower frequencies lie to the left and the higher frequencies lie to the right.

A periodic waveform can be represented by a graph of its values plotted against time, or as a frequency spectrum. They are complementary representations; each stresses some features of the waveform but ignores others. The graph plotted against time shows the value or size of the wave

at each instant in time, but gives no information about how the frequency components of the waveform are distributed. On the other hand, the spectrum gives explicit information about the frequency components but little hint about the shape of the waveform. The two representations are like the two sides of the same coin; neither tells the whole story but together they give two useful views about what is going on.

Activity 29 *Sketching the spectrum*

Look back to Subsection 3.3 to find the Fourier series for the square wave and the sawtooth wave. Using this information, sketch the spectrum of the following periodic waves for frequencies up to 10 kHz:

(a) a square wave with a frequency of 1000 Hz (or 1 kHz);

(b) a sawtooth wave with a period of 1 millisecond (10^{-3} seconds).

In each case, assume that the amplitude of the fundamental is 1.

The sounds produced by musical instruments have complex waveforms. Figure 35 shows again the waveforms produced by a trumpet and a flute sounding the note A at 440 Hz.

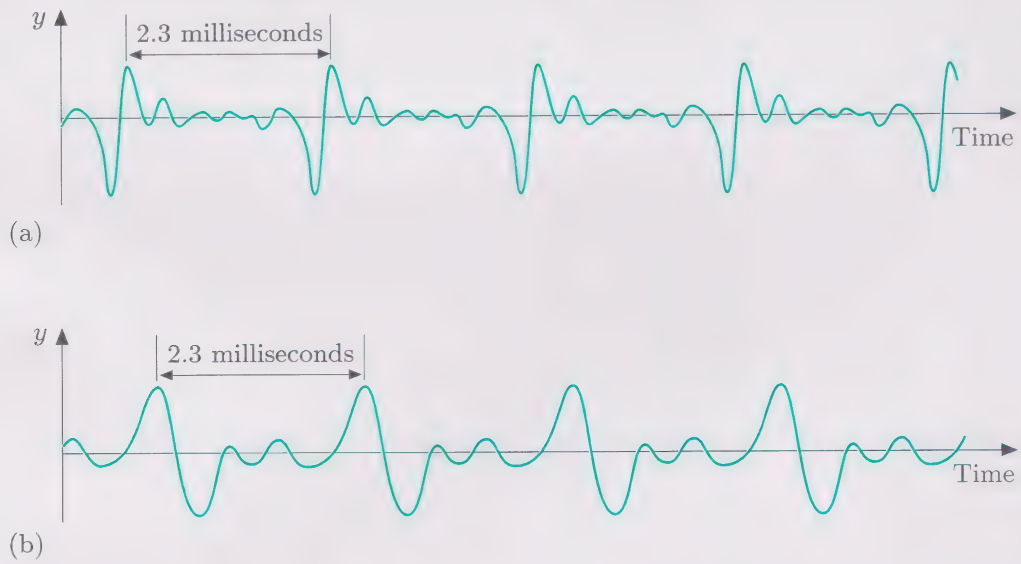


Figure 35 Waveforms produced by (a) a trumpet and (b) a flute

▶ What are the similarities and the differences between the two waveforms?

Both waveforms have the same period of 1/440 seconds, or about 2.3 milliseconds. The flute has a characteristically soft and pure tone and this produces a relatively smoothly-changing waveform. During one period the waveform goes through one large oscillation followed by two smaller ones. There are no sudden changes and the turning points of the waveform

are smooth and rounded. In contrast, the trumpet produces a sound sometimes described as 'bright' or 'brassy'. Its waveform also has one large oscillation during each period, but this is followed by five smaller peaks. The trumpet's waveform changes faster (the slopes are steeper) than the waveform of the flute and the turning points are sharper, especially during the largest oscillation.

► How are these qualitative differences related to the timbre, or quality, of the sounds of the flute and trumpet?

The spectrum, rather than the waveform, provides the link with the perceived quality or timbre of a sound. Figure 36(a) shows the spectrum associated with the trumpet and Figure 36(b) shows the spectrum associated with the flute, both sounding the note A at 440 Hz.

The spectra in Figure 36 were calculated from the actual waveforms of a trumpet and a flute using digital Fourier analysis techniques.

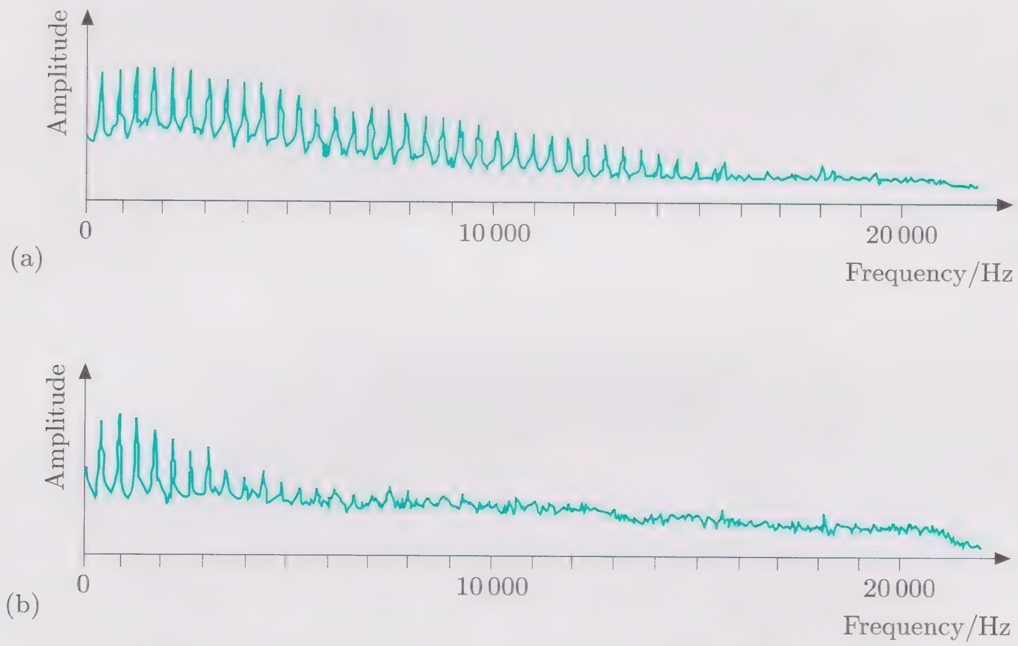


Figure 36 The spectrum of (a) a trumpet and (b) a flute

The frequency scale on the horizontal axis is from 0 Hz to over 20 000 Hz, or 20 kHz. The spectrum of the waveform is represented by the series of peaks or spikes over this frequency range. Each spike represents a particular sine wave frequency component, and the height of each spike represents the amplitude of the component. Note that these plots have been obtained by processing electronically the sounds made by real instruments; they are the results of measurements and calculations on real data, not mathematical idealizations which you saw earlier for the square wave.

Since the trumpet and the flute are sounding the same note—A above middle C—each spectrum will show the same fundamental frequency component at 440 Hz. This is represented by the first spike in each spectrum. The spikes following the fundamental in each case represent the higher frequency harmonics of the sound.

In music, components of a sound that have a higher frequency than the fundamental tone are often called *overtones*. A harmonic is an overtone whose frequency is an integer (whole number) multiple of the fundamental.

Notice the difference between the range of frequencies produced by the two instruments. The harmonics of the flute extend up to about 5000 Hz, after

which it is difficult to distinguish them clearly. The harmonics of the trumpet, however, extend up to at least 15 kHz, three times further than for the flute. Notice also that the height of the spikes in the trumpet's spectrum decreases at a much slower rate than for the flute. Remember that each spike represents a sine-wave component and the height of a spike represents the amplitude of that component. Unlike the flute, the trumpet is a rich source of strong harmonics over a wide range of frequencies.

The human ear is sensitive to frequencies from about 20 Hz to about 20 kHz. This range becomes smaller as people get older.

The differences between the spectra of the trumpet and the flute give direct insight into the quality of the sound produced by each instrument. The note of the flute is relatively pure, and its waveform varies relatively smoothly with time. This is characteristic of a spectrum with relatively few harmonic components at relatively low frequencies.

Contrasted with the soft tones of the flute, the trumpet produces a characteristically harsher or 'brighter' sound, and its waveform contains sharper edges and faster variations. This is characteristic of a spectrum with a wide range of strong harmonics extending to relatively high frequencies.

Pull these ideas together by remembering that the waveform of a sound and the spectrum of a sound are complementary representations. Each gives it own particular stress to the information it is presenting, but also gives hints about what its companion will show.

For the flute, you have seen that smoother waveforms are associated with lower frequencies and a relatively small harmonic range. A sine wave component at a lower frequency is varying more slowly than one at a higher frequency. So a waveform that is changing relatively slowly and smoothly over its period, like that of the flute, is likely to give rise to predominantly lower frequency sine wave components.

For the trumpet, the sharper waveforms are associated with relatively high frequencies and a wide harmonic range. A sine wave at a higher frequency is varying relatively rapidly. So a waveform that contains sharper edges and rapid variations over its period, like that of the trumpet, is likely to contain sine wave components at relatively high frequencies.

What you should take away from this discussion is a broader view of periodic behaviour and an appreciation that not all information about a given situation will be contained in a useful way in just one type of graphical representation or just one mathematical formula. To describe a particular aspect of reality, you may need several different models—each one giving you a different frame within which to picture the world.

If you have time, you should now watch the video 'The sound of silence' again, and complete the following activity.

Activity 30 *Explaining about sounds*

While watching the video, you should pay particular attention to the parts concerning the representation of more complex sounds. As you watch, make some notes to help you with the rest of the activity, starting and stopping the tape as necessary.

Using ideas from the video and the text, make some notes to explain, to someone who is not doing this course, the answers to the following questions.

(a) What is meant by the term 'spectrum'?

(b) Why do the sounds made by a trumpet and a flute both playing the same note have different qualities?

(c) How can two sounds be added together to produce silence?

This section has looked at more complex periodic waves. Although the waves were not simple sine wave shapes, you saw that they could be thought of as being made up of sine wave components. A plot of the amplitudes of the sine wave components against their frequencies gives a frequency spectrum representation of a waveform.

The mix of the sine wave components determines the shape of the waveform. Conversely, features of the waveform determine the shape of the spectrum. In the case of musical sounds, the softer sound of the flute is reflected in its smoothly-varying waveform and in the predominance of lower frequency harmonics in its frequency spectrum. For the trumpet, the sharper waveform of its harsher 'brighter' sound is associated with a wider range of harmonics extending to relatively high frequencies in the audio spectrum.

Periodic waveforms can be represented by plotting a graph of their variation against time, or by plotting the amplitudes of their sine wave components against frequency. The two views are complementary. Each stresses aspects of the periodic behaviour which the other ignores.

Activity 31 *Learning File and Handbook activities*

You should now make sure that you have completed both the Learning File and the Handbook activities for this unit.

Outcomes

After you have studying this section, you should be able to:

- ◇ use the following terms accurately and be able to explain them to someone else: 'sine wave', 'sinusoid', 'beat', 'envelope', 'frequency component', 'fundamental', 'harmonic', 'Fourier series', 'frequency spectrum' (Activities 19 to 22, 27, 29, 30);
- ◇ use your calculator to investigate the beat pattern produced by two sine waves (Activities 20, 22 and 23);
- ◇ explain why a beat pattern can be described either by the sum of two sine waves, or by the product of two different sine waves (Activities 24 and 25);
- ◇ use your calculator to investigate the waveforms associated with the function $A \sin 2\pi ft$, where A can be a constant, a straight-line function or a sinusoidal function (Activity 21);
- ◇ explain how a periodic waveform can be described in terms of a sum of sine waves, or a frequency spectrum (Activity 26, 27, 29 and 30);
- ◇ use your calculator to explore how periodic waveforms can be built up from sine waves (Activities 26, 27 and 28).

Unit summary and outcomes

This unit has been about using sine waves to model periodic behaviour.

Section 1 reviewed some of the mathematical language of sine curves. The amplitude gives the 'height' of a sine curve, measured as the maximum deviation of a peak from the centre line, and the period gives the 'width'—the time for one complete cycle. You saw how a sine curve is generated by circular motion by looking at how the height of one of the pedals on an exercise bike changes as it is pushed round. Plotting the height against time gives a sine curve whose amplitude is equal to the radius of the pedal's circle, and whose period is equal to the time taken for one complete revolution.

Section 2 looked at a sine wave model of the regular change of the time of sunset during a year. You saw how the mathematical curve was matched to the sunset data by choosing appropriately the values of amplitude, frequency or period, phase shift and additive constant. The model was tested by comparing its predictions with the actual data. In any modelling exercise, users must decide whether a model is acceptable for their purpose or whether it should be modified or discarded.

Section 3 looked at some periodic sound patterns. Pure tones sounded together can produce beats, a rhythmic variation in loudness. The video and the unit showed how beats were produced and how they could be modelled mathematically. In more complex periodic patterns, Fourier analysis shows how sine waves are used as building blocks. Each sine wave can be thought of as a frequency component of a waveform and a periodic waveform can be represented in terms of a frequency spectrum of such components. A waveform and its associated spectrum are complementary representations; each stresses aspects which the other ignores.

Sines and cosines are used to model some forms of periodic behaviour on their own, but from a mathematical point of view they are interesting because they can be thought of as the building blocks for more complex periodic functions. Representing periodic behaviour in terms of a spectrum of frequency components offers a new way of thinking about repeating patterns.

Outcomes

Here is a list of the things you should be able to do when you have finished this unit:

- ◇ use the following terms accurately and be able to explain them to someone else: 'periodic behaviour', 'sine curve', 'sine wave', 'sinusoid', 'amplitude, period', 'frequency', 'angular frequency', 'cycles per second', 'hertz', 'radian, phase shift', 'phase lead', 'phase lag', 'identity', 'beat', 'envelope', 'frequency component', 'fundamental', 'harmonic', 'Fourier series', 'frequency spectrum';
- ◇ explain in your own words to someone not taking the course the relationship between circular motion and a sine curve;
- ◇ explain and use the mathematical relationships between frequency, angular frequency and period;
- ◇ write down the formula for a sine curve, given the amplitude and the frequency or period;
- ◇ outline the modelling steps you would take to fit a sine curve to a set of periodic data;
- ◇ relate the sine and cosine functions, using standard trigonometric identities;
- ◇ explain how to translate a sine or cosine curve vertically or horizontally, by adding an appropriate constant or phase shift;
- ◇ interpret and evaluate an expression of the form $M + A\sin(\omega t + \phi)$;
- ◇ explain why a beat pattern can be described either by the sum of two sine waves, or by the product of two different sine waves;
- ◇ explain how a periodic waveform can be described in terms of a sum of sine waves, or a frequency spectrum.

You should also be able to use your calculator to:

- ◇ investigate the beat pattern produced by two sine waves;
- ◇ investigate the waveforms associated with the function $A\sin 2\pi ft$, where A can be a constant, a straight-line function or a sinusoidal function;
- ◇ choose values for the amplitude, period or frequency and phase shift of a sine curve model, given a set of periodic data;
- ◇ explore how periodic waveforms can be built up from sine waves;
- ◇ explore parametric graphing.

Comments on Activities

Activity 1

It can be helpful to think about why you choose to complete particular activities or pieces of work. Sometimes you may think that time can be used more effectively if you are consciously aware that a particular idea is difficult and you need to do further work on it; sometimes time may not be well spent by completing activities you know you can do well.

This unit makes use of ideas and terms that have been introduced earlier in the course so the activities provide a useful tool for checking what you know and integrating new ideas.

Activity 2

The parameters of the sine waves are:

(a) amplitude: 1, period: 1 second, frequency: 1 Hz;

(b) amplitude: 0.5, period: 4 seconds, frequency: 0.25 Hz;

(c) amplitude: 50, period: 0.02 seconds, frequency: 50 Hz.

Activity 3

A quarter of a circle (90 degrees) corresponds to $2\pi/4 = \pi/2$ radians. Halfway round the circle (180 degrees) is $2\pi/2 = \pi$ radians, and three-quarters of a circle (270 degrees) is $3/4 \times 2\pi = 3\pi/2$ radians.

Activity 4

As x changes from 0 to 2π, the value of t on the vertical axis changes in direct proportion from 0 to T. The quantity m represents the slope of the graph, so $m = T/2\pi$.

The relationship between t and x is $t = (T/2\pi)x$. To undo the multiplication by $T/2\pi$, multiply both sides by $(2\pi/T)$, the result is:

$$\left(\frac{2\pi}{T}\right)t = \left(\frac{2\pi}{T}\right) \times \left(\frac{T}{2\pi}\right)x = x$$

so:

$$x = \left(\frac{2\pi}{T}\right)t$$

Activity 5

(a) The formula describing the sine curve is $0.1\sin(2\pi \times 3 \times t)$, or $0.1\sin(6\pi t)$.

(b) The period T of the curve is the reciprocal of the frequency in Hz, so $T = 1/3$ seconds.

(c) The first peak of a sine curve occurs a quarter of a period after $t = 0$, at $t = (1/3)/4 = 1/12$ seconds, or 0.0833 seconds.

Activity 6

(a) The amplitude is 0.001 m or 1 mm.
The needle will have travelled through one complete cycle when $30t = 2\pi$. This occurs at a time $t = 2\pi/30 = 0.21$ seconds.

(b) Two complete cycles will take 0.42 seconds.

(c) For the same amplitude, the formula for the needle's position may be expressed as $y = 0.001\sin\omega t$.

The period T of the motion is 0.125 seconds, so the angular frequency is given by $\omega = 2\pi/T = 2\pi/0.125 \simeq 50.3$ radians per second and the formula is:

$$y = 0.001\sin 50.3t$$

Activity 7

Read the summary after the activity and check you have included all the points.

Activity 8

The period T is one year, or 52 weeks.

From midsummer to midwinter the time of sunset varies from a maximum of 20.22 GMT to a minimum of 15.52 GMT, a total range of 4.5 hours. A suitable value for the amplitude is half this value, 2.25 hours.

Activity 9

There are no comments for this activity.

Activity 10

(a) Over the interval $0 \leq x \leq 2\pi$, the value of $\sin(x + \pi/2)$ is equal to zero for $x = \pi/2$ and $3\pi/2$. The function is equal to -1 for $x = \pi$, and equal to $+1$ for $x = 0$ and 2π.

(b) The two curves are shifted relative to each other by a phase shift of $\pi/2$ radians. The curve of $\sin x$ lags the curve for $\sin(x + \pi/2)$; alternatively $\sin(x + \pi/2)$ leads $\sin x$.

Activity 11

The model is:

$$\text{sunset time} = 18.12 + 2.25 \sin\left(\frac{2\pi}{52}t - 1.21\right)$$

Putting $t = 43$ gives :

$$\begin{aligned}\text{sunset time} &= 18.12 \\ &\quad + 2.25 \sin\left(\frac{2\pi}{52} \times 43 - 1.21\right) \\ &= 18.12 + 2.25 \sin(5.20 - 1.21) \\ &= 18.12 + 2.25 \sin 3.99 \\ &\doteq 18.12 - 1.68 \\ &= 16.44, \text{ or } 16.26\,\text{GMT}\end{aligned}$$

The actual sunset time for the start of the week in which Guy Fawkes' Night falls is 16.31. The difference between the data and the model's predictions in this case is only 5 minutes.

Activity 12

Here are some ideas. One observation is that the sunset data is essentially discrete, the sun sets only once each day. There is no 'sunset curve' (whether based on weekly or daily data), only a set of numbers. Plotting these data and then 'seeing' a curve is a modelling assumption. The sine curve model is a continuous model chosen because its overall shape is similar to the 'sunset curve'. It can in principle be used to predict the time of sunset for *any* value of t, not just the integer values $0, 1, 2, \ldots, 51$.

The model also assumes that the shape of the sunset curve will always be the same, so it would not be useful for very long time predictions over thousands of years, because it ignores gradual changes in the Earth's orbit.

What constitutes a good fit in a model depends on what you want the model for, and how numerically accurate you want it to be. There are various mathematical techniques which define goodness of fit in particular ways. These work to adjust the parameters of the proposed model so that the 'best' fit using a particular criterion is found. Depending on your requirements, you may want a model whose predictions all fall within a certain percentage of the actual data, or whose predictions on average do not fall outside of a certain range.

Activity 13

The cosine function produces a curve which is identical to the sine curve—*except* that it is shifted in phase relative to it by a quarter of a period, or $\pi/2$ radians.

The only effect of this phase shift is that the cosine curve starts at $+1$ at $x = 0$, instead of at 0. Otherwise the sine and cosine curves share the same periodic shape, the same amplitude (equal to 1), and the same period of 2π radians.

Activity 14

The $\pi/2$ phase shift effectively shifts the $\cos x$ curve to the left along the x-axis. As a result, the $\cos(x + \pi/2)$ curve is identical to the curve of $-\sin x$, and you can write

$$\cos(x + \pi/2) = -\sin x$$

You can check this identity for particular values of x. For example, if $x = 0$, then $\cos(\pi/2) = 0$, which is the same as $-\sin 0 = 0$. If $x = 1$, then $\cos(1 + \pi/2) = \cos 2.5708 = -0.8415$, which is the same as $-\sin 1 = -0.8415$. Try some other values of x for yourself.

(You might have a different identity; for example, $\cos(x + \pi/2) = \sin(x + \pi)$.)

Activity 15

Function	Identity	Comment
$\sin(x - \pi/2)$	$-\cos x$	The sine curve is shifted $\pi/2$ radians to the right
$\cos(x + 2\pi)$	$\cos x$	The cosine curve is shifted 2π radians to the left
$\sin(x - \pi)$	$-\sin x$	The curve is shifted π radians to the right
$\cos(x + 3\pi/2)$	$\sin x$	The curve is shifted $3\pi/2$ radians to the left

Activity 16

If x represents the expression $((2\pi/52)t - 1.21)$, the model can be written as:

$$18.12 + 2.25\sin(x)$$

Activity 15 included the identity $\sin x = \cos(x + 3\pi/2)$. Using this identity the model can be rewritten as

$$18.12 + 2.25\cos\left(x + \frac{3\pi}{2}\right)$$

But $x + 3\pi/2 = (2\pi/52)t - 1.21 + 3\pi/2$ $\simeq (2\pi/52)t + 3.50$, so the model becomes:

$$18.12 + 2.25\cos\left(\frac{2\pi}{52}t + 3.50\right)$$

Check for yourself that the models give the same results (to 2 decimal places) for different values of t.

Activity 17

Over any range of values of x, the sum of $\sin^2 x$ and $\cos^2 x$ is a horizontal straight line of constant height 1.

Activity 18

There are no comments on this activity.

Activity 19

(a) As a point moves round a circle, its vertical height above the centre traces out a sine curve. The radius of the circle determines the maximum height: this is the amplitude of the sine curve. The time to go round the circle once corresponds to one cycle of the sine function: this is the period. The frequency is the number of times the point goes round the circle in unit time corresponding to the frequency of oscillation of the sine curve. The phase shift corresponds to where on the circle the point starts or the horizontal translation of the sine curve from the standard sine curve.

(b) A beat is produced when two sine functions, or notes, very close in frequency and of similar amplitude are added together. The resulting waveform has oscillations whose amplitude (loudness) varies periodically.

Activity 20

Tones of 6 Hz and 8 Hz together produce a beat of 2 Hz, equal to their frequency difference. The period of the beat is $1/2 = 0.5$ seconds.

Activity 21

The formula for a straight line is $y = mt + c$, where m is the gradient and c is the intercept on the y-axis. t represents the time variable plotted along the horizontal axis.

A line with a slope of 0.5 and an intercept of 1 is described by the formula $y = 0.5t + 1$.

The growing sine curve with a frequency of 1 Hz has the formula $y = (0.5t + 1)\sin(2\pi t)$, and is shown in Figure 37(a).

If the slope of the line is -0.5 and the frequency is 1 Hz, the formula is $y = (-0.5t + 1)\sin(2\pi t)$. The straight-line envelope will cross the horizontal axis at $t = 2$. The amplitude of the envelope will be 1 at $t = 0$, reducing to zero at $t = 2$ and showing growth without limit after $t = 2$. Figure 37(b) shows the waveform.

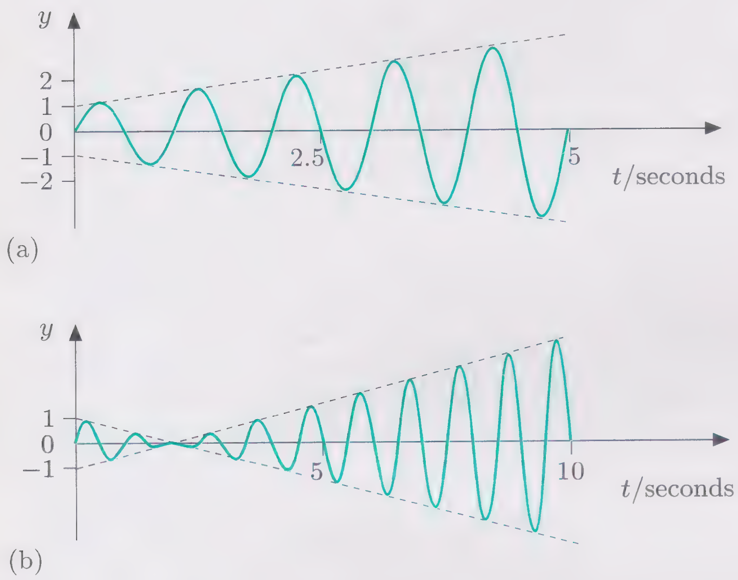


Figure 37 Plot of the functions (a) $(0.5t + 1)\sin(2\pi t)$ and (b) $(-0.5t + 1)\sin(2\pi t)$

Activity 22

You should see a periodic beat pattern the envelope of which reaches a maximum value of 2 at the times $t = 0$, $t = 1$ and $t = 2$ seconds, and a minimum value of 0 at $t = 0.5$ and $t = 1.5$ seconds.

The pattern repeats with a period of 1 second so the beat frequency is 1 Hz, equal to the frequency difference between the 6 Hz and 7 Hz sine waves.

The cosine frequency $2\cos 2\pi\left(\dfrac{7-6}{2}\right)t = 2\cos \pi t$ defines the envelope of the beat pattern. Note that the beat frequency of 1 Hz is twice the frequency of the cosine envelope.

The beat frequency for the 12 Hz and 16 Hz tones is 4 Hz. The formula of the cosine envelope function is $2\cos 2\pi\left(\dfrac{16-12}{2}\right)t = 2\cos 4\pi t$. The beat frequency is twice the frequency of the envelope, 2 Hz in this case, because there are two amplitude peaks in each cycle of the envelope.

Activity 23

For frequencies of 6 Hz and 7 Hz, you should find that the zero-crossings of the wave inside the envelope occur at intervals of about 0.154 seconds, corresponding to a frequency of about 6.5 Hz.

For 9 Hz and 11 Hz, the zero-crossing occur at intervals of about 0.1 seconds, corresponding to a frequency of 10 Hz.

For 14 Hz and 18 Hz, the zero-crossing intervals are about 0.0625 seconds, corresponding to a frequency of 16 Hz.

The pattern is that the frequency of the wave inside the beat envelope is the arithmetic mean of the two beating frequencies.

Activity 24

(a) Putting $t = 1$ second and $f_1 = f_2 = 1$ Hz gives:

$$\sin 2\pi + \sin 2\pi = 2\cos 0 \sin 2\pi$$

Now $\cos 0 = 1$, so the identity reduces to $2\sin 2\pi = 2\sin 2\pi$, or $0 = 0$. So the identity is true in this case.

(b) Putting $t = 0$, $f_1 = 0$ Hz, $f_2 = 1$ Hz gives:

$$\sin 0 + \sin 0 = 2\cos 0 \sin 0$$

Since $\cos 0 = 1$, this reduces to $2\sin 0 = 2\sin 0$, or $0 = 0$. The identity is true for these values of t, f_1 and f_2.

(c) Putting $t = t$ and $f_1 = f_2 = f$ gives

$$\sin 2\pi ft + \sin 2\pi ft = 2\cos 0 \sin 2\pi ft$$

Again $\cos 0 = 1$, so the identity reduces to $2\sin 2\pi ft = 2\sin 2\pi ft$, which is true.

Activity 25

(a) Replacing $2\pi f_1$ by ω_1 and $2\pi f_2$ by ω_2 in the identity, gives a rather neater form:

$$\begin{aligned}&\sin\omega_1 t + \sin\omega_2 t\\ &= 2\cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t \sin 2\pi\left(\frac{f_1 + f_2}{2}\right)t\\ &= 2\cos\left(\frac{2\pi f_1 - 2\pi f_2}{2}\right)t \sin\left(\frac{2\pi f_1 + 2\pi f_2}{2}\right)t\\ &= 2\cos\left(\frac{\omega_1 - \omega_2}{2}\right)t \sin\left(\frac{\omega_1 + \omega_2}{2}\right)t\end{aligned}$$

(b) In the most general form, the products $\omega_1 t$ and $\omega_2 t$ are replaced by single symbols a and b. Putting $a = \omega_1 t$ and $b = \omega_2 t$, the identity takes the form you are most likely to see if you looked it up in a mathematics book:

$$\sin a + \sin b = 2\cos\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right)$$

Activity 26

(a) The frequency f_1 of the fundamental is the same as the frequency of the square wave, so $f_1 = 1/0.5 = 2\,\text{Hz}$. The frequencies of the harmonics are odd multiples of the fundamental; $3 \times 2 = 6\,\text{Hz}$, $5 \times 2 = 10\,\text{Hz}$, $7 \times 2 = 14\,\text{Hz}$, and so on.

(b) The Fourier series of the square wave is:

$$\sin 4\pi t + \frac{1}{3}\sin 12\pi t + \frac{1}{5}\sin 20\pi t + \frac{1}{7}\sin 28\pi t + \cdots$$

where the terms 4π, 12π, 20π, and so on, are the angular frequencies of the fundamental, third harmonic, fifth harmonic, ... expressed in radians per second.

(c) You should see the waveform becoming less like a sine wave and closer to a square wave.

Activity 27

The period T_1 of the sawtooth wave is $2\pi/\omega_1 = 2\pi/20 = 0.3142$ seconds, corresponding to a fundamental frequency of 3.18 Hz. The Fourier series is:

$$\sin 20t + \frac{1}{2}\sin 40t + \frac{1}{3}\sin 60t + \frac{1}{4}\sin 80t + \frac{1}{5}\sin 100t + \cdots$$

Activity 28

The Fourier series of the phase shifted sawtooth wave is:

$$\sin(20t + \pi) + \frac{1}{2}\sin(40t + \pi) + \frac{1}{3}\sin(60t + \pi) + \frac{1}{4}\sin(80t + \pi) + \cdots$$

When you display this series you should see the original sawtooth wave upside down.

The result of adding the original and the phase shifted waveforms is zero.

Activity 29

(a) A square wave with a frequency of 1000 Hz (or 1 kHz) will have a fundamental component at 1000 Hz and harmonics at odd multiples of the fundamental at 3 kHz, 5 kHz, 7 kHz, 9 kHz, and so on. Figure 38(a) shows the spectrum of the square wave up to 10 kHz (10 000 Hz). Notice how the amplitude of the harmonic frequency components gets smaller as the frequency increases.

(b) The fundamental frequency of the sawtooth wave is $1/0.001 = 1000\,\text{Hz} = 1\,\text{kHz}$. The sawtooth has frequency components at the fundamental and its integer multiples, Figure 38(b) shows the spectrum. As with the square wave, the amplitudes of the harmonics decrease with frequency, but they never completely disappear.

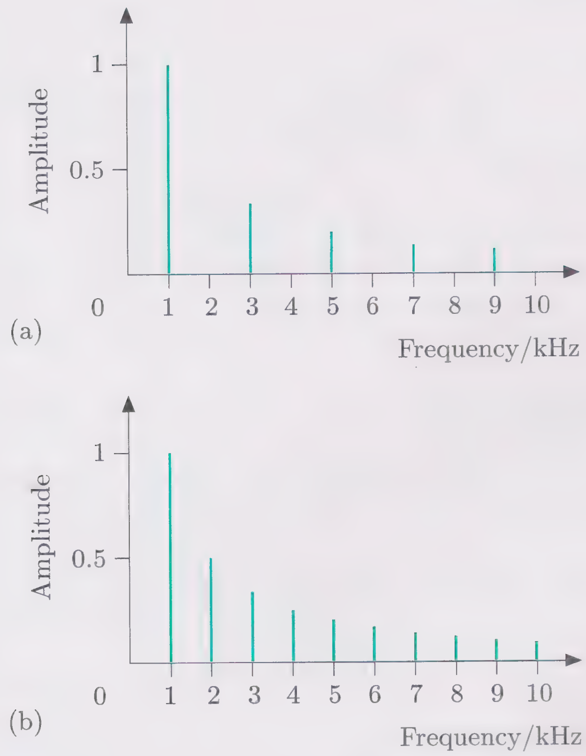


Figure 38 Amplitude spectra for (a) a square wave and (b) a sawtooth wave

Activity 30

Your answers will be personal to you.

Writing is a highly complex activity (MU120 units went through numerous drafts before the final version you see) at which many people can and need to become proficient. Just as each person's speech is unique, so no two people write exactly alike. Writing is an expression of individuality, as well as an attempt to communicate with other people.

When people speak face to face with other people, they are usually able to get rapid feedback because they see and hear their audience react.

However, when you write, the audience is more distant, or you may be writing for yourself. Because of this lack of immediate feedback when you are producing a piece of written material, it is important to think of the audience in a conscious way.

What role are you playing? As a writer, expert, friend or colleague?

What role do you expect the audience to play? As the general public, informed expert, or friend?

The material you produce will be expressed differently depending on these roles. One way to build a sense of audience into your written work is to consider two main questions:

How well do you know the people for whom you are writing, and how well do they know you?

How much do you think they will know about the subject?

It is also a good idea to ask yourself:

Why write?

Who will read it?

What format should it take?

The process of thinking through these questions will also help to determine the general tone of what you produce. Try to keep your written work focused on the purpose and the audience. In this way, you should produce written material of high quality.

Activity 31

There are no comments for this activity.

Acknowledgements

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